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The Influence of Satellites upon the Form of Saturn's Ring

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IV. *The Influence of Satellites upon the Form of Saturn's Ring.*

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§ 1. *Introduction.*

IN his "Adams' Prize Essay"* for the year 1856, MAXWELL showed that the rings of the planet Saturn could only be stable for small disturbances on the theory that they were composed of meteorites sufficiently small. This has been confirmed since by spectroscopic evidence and is now generally accepted. In continuance of the same idea, the various divisions of the rings have been accounted for by presuming that, in those positions where a single particle moving in a circular orbit about the planet would have a period simply commensurate with that of one of the nearer satellites of Saturn, instability would result. This idea has been fully emphasized recently by LOWELL.† His observations at Flagstaff have disclosed a large number of additional divisions in the rings (see Appendix to this paper). They have the appearance of fine lines traced on the surface of the rings. In each case LOWELL is able to show that the divisions occur at intervals of periods commensurable with that of satellite Mimas. The periods have the ratios such as $\frac{3}{5}$, $\frac{6}{11}$, $\frac{1}{2}$, $\frac{5}{11}$, &c. LOWELL has stated the argument for this view in 'Bulletin,' 32, p. 189. If the action of one body upon another revolving about a third be examined by the method of the variation of arbitrary constants, in the expressions for the periodic inequalities in the radius vector and the longitude, there appear terms of the type $[C/(pn-qn')]\cos\{(pn-qn')t+Q\}$, where n and n' are the mean motions of the perturbing and perturbed bodies, p and q are integers, and the remaining quantities are constants. It is clear that when the ratio n/n' is approximately equal to q/p, then the inequality will become very large.

We may take a satellite of Saturn as one of the bodies and one of the particles forming the ring as the other; if $n/n' = q/p$, approximately, then the particle will

* MAXWELL'S 'Collected Works,' I., p. 288.

† LOWELL, 'Observatory Bulletin,' No. 66.

depart considerably from its unperturbed path and collision with other particles will result. In this way the divisions in the ring have been explained.

Some doubt has been cast upon this theory, and it has been shown* that even when n and n' are commensurable, a closer examination of the motion leads to the conclusion that the denominator will not vanish.

It is also noticeable that this explanation takes no account of the attraction of the numerous particles upon one another, which may be considerable.

A re-examination of the matter is made in the present paper. As the satellites of Saturn are all approximately in the same plane as the ring, the problem is formulated in two dimensions only. The satellite is assumed to follow an unperturbed circular orbit, and the problem reduces to a slight variation of the "restricted problem" of three bodies. We shall consider the effect of this satellite upon a number of particles forming a single ring round the planet, subject to their mutual attraction as well as that of the satellite and of Saturn. The actual Saturnian rings are supposed to be composed of a number of such rings arranged concentrically. These will have some effect one upon the other, but, for the present, this effect is disregarded.

In his paper, MAXWELL considered the single ring of particles only. He found that the equations of motion could be satisfied by assuming that the particles rotated round the primary in a circle with suitable angular motion. He then examined the effect of a small arbitrary disturbance upon them, and his results show that the disturbances would remain small if the masses of the particles were sufficiently small. That is, the ring would be "ordinarily" stable.

In the present paper the plan is different. The disturbance of the ring of particles by the satellite is examined, with a view to determining under what conditions the departure from a certain fixed circle will be large. It is clear that if the departures do become large, collisions with adjacent rings of particles will result, and the particles will leave the vicinity of the original circle irrevocably. In this case a division in the ring will result. It is with this meaning that the terms stability and instability have been used in the paper. But, as will be pointed out again in its proper place, the orbits in which the departure from the circular form does not become great with increase of time may yet become "ordinarily" unstable if further small arbitrary displacements are imposed upon them.

The results of this paper will therefore indicate some, but not necessarily all, of the positions of divisions in the rings due to instability of whatever kind.

In §§ 2 to 4 an analytical theory is fully worked out on the supposition of equal particles in each ring. In § 5 it is shown how amendments may be introduced to cover the case of unequal particles. The application to the Saturnian system is given in § 6, and the last paragraph summarises the results obtained.

* TISSERAND, 'Méc. Céleste,' vol. iv., p. 420.

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§ 2. *Formation of the Equations.*

Let M be the mass of the primary and m' the mass of the principal satellite which is assumed to describe an unperturbed circle round the primary. Take the origin at M. Let there be n particles forming a ring round the primary, subject to attraction from M, m', and one another, and let the mass and co-ordinates at time t of particle λ be m_λ, r_λ, θ_λ. If the co-ordinates of m' at the same time are r', θ', then the motion of particle λ will be produced by forces which are the derivatives of the function

$$\mathrm{F} \equiv \frac{\mathrm{M}+m_\lambda}{r_\lambda}+\frac{m'}{\Delta_\lambda}-\frac{m'r_\lambda}{r'^2}\cos(\theta'-\theta_\lambda)+\sum_\mu \frac{m_\mu}{\mathrm{D}_{\lambda\mu}} - \sum_\mu \frac{m_\mu r_\lambda}{r_\mu^2}\cos(\theta_\mu-\theta_\lambda);$$

where

$$\Delta_\lambda^2 = r'^2+r_\lambda^2-2r'r_\lambda\cos(\theta'-\theta_\lambda),$$

and

$$\mathrm{D}_{\lambda\mu}^2 = r_\mu^2+r_\lambda^2-2r_\mu r_\lambda\cos(\theta_\mu-\theta_\lambda).$$

The equations of motion of m_λ are then

$$\left.\begin{aligned} \frac{d^2r_\lambda}{dt^2}-r_\lambda\left(\frac{d\theta_\lambda}{dt}\right)^2 &= \frac{\partial \mathrm{F}}{\partial r_\lambda}, \\ \frac{1}{r_\lambda}\frac{d}{dt}(r_\lambda^2\dot{\theta}_\lambda) &= \frac{1}{r_\lambda}\frac{\partial \mathrm{F}}{\partial \theta_\lambda}. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

As we are assuming that m' describes an unperturbed circle,

$$r' = a' \qquad \text{and} \qquad \theta' = \omega' t+\epsilon',$$

where $\omega'^2a'^3 = \mathrm{M}+m' = \mathrm{M}$, with sufficient approximation.

Let us assume now that the remaining particles are moving in the vicinity of the vertices of a regular polygon of radius a. Then we may put

$$r_\lambda = a+\rho_\lambda,$$

$$\theta_\lambda = \omega t+\epsilon+\lambda\,.\,2\pi/n+\sigma_\lambda,$$

for all values of λ from 1 to n, where ρ and σ are assumed small, so that squares, products, and higher powers of them and their first derivatives with regard to the time may be neglected.

The equations (1) now reduce to

$$\left.\begin{aligned}\frac{d^2}{dt^2}\rho_\lambda-2a\omega\frac{d}{dt}\sigma_\lambda-\omega^2\rho_\lambda-a\omega^2 &= \left(\frac{\partial \mathbf{F}}{\partial r_\lambda}\right)_0+\sum_\mu\rho_\mu\left(\frac{\partial^2\mathbf{F}}{\partial r_\mu\partial r_\lambda}\right)_0\\ &\quad+\sum_\mu\sigma_\mu\left(\frac{\partial^2\mathbf{F}}{\partial\theta_\mu\partial r_\lambda}\right)_0,\\ a\frac{d^2\sigma_\lambda}{dt^2}+2\omega\frac{d\rho_\lambda}{dt} &= \left(\frac{1}{r_\lambda}\frac{\partial\mathbf{F}}{\partial\theta_\lambda}\right)_0+\sum_\mu\rho_\mu\left(\frac{\partial^2\mathbf{F}}{r_\lambda\partial r_\mu\partial\theta_\lambda}\right)_0\\ &\quad+\sum_\mu\sigma_\mu\left(\frac{\partial^2\mathbf{F}}{r_\lambda\partial\theta_\mu\partial\theta_\lambda}\right)_0.\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

In order to determine the derivatives, we write in the formula for Δ_λ, $\phi_\lambda = \theta'-\theta_\lambda$, and $\alpha = r_\lambda/r'$. Then

$$\begin{aligned}\Delta_\lambda^{-1} &= \{1+\alpha^2-2\alpha\cos\phi_\lambda\}^{-\frac{1}{2}}\div r'\\ &= \{\tfrac{1}{2}b_0+b_1\cos\phi+\ldots+b_i\cos i\phi+\ldots\}\div r'\end{aligned}$$

by FOURIER'S series.

This series will be taken as absolutely and uniformly convergent.

We find then

$$\frac{\partial\mathbf{F}}{\partial r_\lambda} = -\frac{\mathrm{M}+m_\lambda}{r_\lambda^{\ 2}}+\frac{m'}{r'^2}\left\{\frac{1}{2}\frac{\partial b_0}{\partial\alpha}+\ldots+\frac{\partial b_i}{\partial\alpha}\cos i\phi+\ldots\right\}-\frac{m'}{r'^2}\cos\phi$$
$$-\sum_\mu m_\mu\left\{\frac{r_\lambda-r_\mu\cos(\theta_\mu-\theta_\lambda)}{\mathrm{D}_{\lambda\mu}^{\ 3}}+\frac{\cos(\theta_\mu-\theta_\lambda)}{r_\mu^{\ 2}}\right\},$$

$$\sum_\mu\rho_\mu\frac{\partial^2\mathbf{F}}{\partial r_\lambda\partial r_\mu} = \left[2\frac{\mathrm{M}+m_\lambda}{r_\lambda^{\ 3}}+\frac{m'}{r'^3}\frac{\partial^2}{\partial\alpha^2}(\tfrac{1}{2}b_0+\ldots+b_i\cos i\phi\ldots)\right.$$
$$\left.-\sum_\mu m_\mu\left\{\frac{1}{\mathrm{D}_{\lambda\mu}^{\ 3}}-\frac{3(r_\lambda-r_\mu\cos(\theta_\mu-\theta_\lambda))^2}{\mathrm{D}_{\lambda\mu}^{\ 5}}\right\}\right]\rho_\lambda$$
$$+\sum_\mu m_\mu\left[\frac{\cos(\theta_\mu-\theta_\lambda)}{\mathrm{D}_{\lambda\mu}^{\ 3}}+\frac{3\{r_\lambda-r_\mu\cos(\theta_\mu-\theta_\lambda)\}\{r_\mu-r_\lambda\cos(\theta_\mu-\theta_\lambda)\}}{\mathrm{D}_{\lambda\mu}^{\ 5}}\right.$$
$$\left.+\frac{2\cos(\theta_\mu-\theta_\lambda)}{r_\mu^{\ 3}}\right]\rho_\mu,$$

$$\sum_\mu\sigma_\mu\frac{\partial^2\mathbf{F}}{\partial\theta_\mu\partial r_\lambda} = \left[\frac{m'}{r'^2}\frac{\partial}{\partial\alpha}(b_1\sin\phi+\ldots+ib_i\sin i\phi+\ldots)-\frac{m'}{r'^2}\sin\phi\right]\sigma_\lambda$$
$$-\sum_\mu m_\mu\left[\frac{r_\mu\sin(\theta_\mu-\theta_\lambda)}{\mathrm{D}_{\mu\lambda}^{\ 3}}-\frac{3\{r_\lambda-r_\mu\cos(\theta_\mu-\theta_\lambda)\}r_\lambda r_\mu\sin(\theta_\mu-\theta_\lambda)}{\mathrm{D}_{\lambda\mu}^{\ 5}}\right.$$
$$\left.-\frac{\sin(\theta_\mu-\theta_\lambda)}{r_\mu^{\ 2}}\right](\sigma_\mu-\sigma_\lambda),$$

$$\frac{1}{r_\lambda}\frac{\partial\mathbf{F}}{\partial\theta_\lambda} = \frac{m'}{r'r_\lambda}(\ldots+ib_i\sin i\phi+\ldots)-\frac{m'}{r'^2}\sin\phi+\sum_\mu m_\mu\left[\frac{r_\mu\sin(\theta_\mu-\theta_\lambda)}{\mathrm{D}_{\lambda\mu}^{\ 3}}-\frac{1}{r_\mu^{\ 2}}\sin(\theta_\mu-\theta_\lambda)\right],$$

$$\Sigma_{\mu}\rho_{\mu}\frac{\partial^2\mathbf{F}}{r_{\lambda}\partial r_{\mu}\partial\theta_{\lambda}} = \left[\frac{m'}{r'^2 r_{\lambda}}\frac{\partial}{\partial a}(b_1\sin\phi+\ldots+ib_i\sin i\phi+\ldots)\right.$$
$$-\frac{m'}{r'r_{\lambda}^2}(\ldots+ib_i\sin i\phi\ldots)$$
$$\left.-\Sigma_{\mu}m_{\mu}\left\{\frac{3r_{\mu}\sin(\theta_{\mu}-\theta_{\lambda})\{r_{\lambda}-r_{\mu}\cos(\theta_{\mu}-\theta_{\lambda})\}}{\mathrm{D}_{\mu\lambda}^5}\right\}\right]\rho_{\lambda}$$
$$+\Sigma_{\mu}m_{\mu}\left[\frac{\sin(\theta_{\mu}-\theta_{\lambda})}{\mathrm{D}_{\lambda\mu}^3}-\frac{3r_{\mu}\sin(\theta_{\mu}-\theta_{\lambda})\{r_{\mu}-r_{\lambda}\cos(\theta_{\mu}-\theta_{\lambda})\}}{\mathrm{D}_{\lambda\mu}^5}\right.$$
$$\left.+\frac{2}{r_{\mu}^3}\sin(\theta_{\mu}-\theta_{\lambda})\right]\rho_{\mu},$$

$$\Sigma\sigma_{\mu}\frac{\partial^2\mathbf{F}}{r_{\lambda}\partial\theta_{\mu}\partial\theta_{\lambda}} = \left[-\frac{m'}{r'r_{\lambda}}(\ldots i^2b_i\cos i\phi\ldots)+\frac{m'}{r'^2}\cos\phi\right]\sigma_{\lambda}$$
$$+\Sigma_{\mu}m_{\mu}\left[\frac{r_{\mu}\cos(\theta_{\mu}-\theta_{\lambda})}{\mathrm{D}_{\mu\lambda}^3}-\frac{3r_{\mu}^2r_{\lambda}\sin^2(\theta_{\mu}-\theta_{\lambda})}{\mathrm{D}_{\lambda\mu}^5}\right.$$
$$\left.-\frac{1}{r_{\mu}^2}\cos(\theta_{\mu}-\theta_{\lambda})\right](\sigma_{\mu}-\sigma_{\lambda}).$$

In the summations of the right-hand members, μ takes all integral values from 1 to n, except $\mu=\lambda$.

The zero values of these derivatives are obtained by putting

$$r'=a', \qquad r_{\lambda}=a,$$
$$\theta'=\omega't+\epsilon', \qquad \theta_{\lambda}=\omega t+\epsilon+\lambda 2\pi/n.$$

Whence

$$\Delta_{\lambda}^2=a'^2+a^2-2aa'\cos\phi$$

where ϕ now is

$$(\omega'-\omega)t+\epsilon'-\epsilon-\lambda 2\pi/n;$$

and

$$\mathrm{D}_{\lambda\mu}=2a\sin(\mu-\lambda)\pi/n.$$

Then

$$(\partial\mathbf{F}/\partial r_{\lambda})_0=-\frac{\mathrm{M}+m_{\lambda}}{a^2}+\frac{m'}{a'^2}\left(\tfrac{1}{2}\frac{db_0}{da}+\ldots+\frac{db_i}{da}\cos i\phi+\ldots\right)-\frac{m'}{a'^2}\cos\phi$$
$$-\Sigma_{\mu}m_{\mu}\left\{\frac{1}{4a^2\sin(\mu-\lambda)\pi/n}+\frac{1}{a^2}\cos(\mu-\lambda)2\pi/n\right\},$$

$$\Sigma\rho_{\mu}\left(\frac{\partial^2\mathbf{F}}{\partial r_{\lambda}\partial r_{\mu}}\right)_0=\left[\frac{2(\mathrm{M}+m_{\lambda})}{a^3}+\frac{m'}{a'^3}\frac{\partial^2}{\partial a^2}(\tfrac{1}{2}b_0+\ldots+b_i\cos i\phi+\ldots)\right.$$
$$\left.-\Sigma_{\mu}m_{\mu}\left\{\frac{1}{8a^3\sin^3(\mu-\lambda)\pi/n}-\frac{3}{8a^3\sin(\mu-\lambda)\pi/n}\right\}\right]\rho_{\lambda}$$
$$+\Sigma_{\mu}m_{\mu}\left[\frac{\cos(\mu-\lambda)2\pi/n}{8a^3\sin^3(\mu-\lambda)\pi/n}+\frac{3}{8a^3\sin(\mu-\lambda)\pi/n}+\frac{2\cos(\mu-\lambda)2\pi/n}{a^3}\right]\rho_{\mu},$$

$$\Sigma\sigma_\mu\left(\frac{\partial^2\mathbf{F}}{\partial\theta_\mu\partial r_\lambda}\right)_0 = \left[\frac{m'}{a'^2}\frac{\partial}{\partial\alpha}(b_1\sin\phi+\ldots+ib_i\sin i\phi\ldots)-\frac{m'}{a'^2}\sin\phi\right]\sigma_\lambda$$
$$+\sum_\mu m_\mu\left[\frac{\delta\cos(\mu-\lambda)\,\pi/n}{8a^2\sin^2(\mu-\lambda)\,\pi/n}+\frac{\sin(\mu-\lambda)\,2\pi/n}{a^2}\right](\sigma_\mu-\sigma_\lambda),$$

$$\left(\frac{1}{r_\lambda}\frac{\partial\mathbf{F}}{\partial\theta_\lambda}\right)_0 = \frac{m'}{a'a}(\ldots+ib_i\sin i\phi\ldots)-\frac{m'}{a'^2}\sin\phi$$
$$+\sum_\mu m_\mu\left[\frac{\cos(\mu-\lambda)\,\pi/n}{4a^2\sin^2(\mu-\lambda)\,\pi/n}-\frac{1}{a^2}\sin(\mu-\lambda)\,2\pi/n\right],$$

$$\sum_\mu\rho_\mu\left(\frac{\partial^2\mathbf{F}}{r_\lambda\partial r_\mu\partial\theta_\mu}\right)_0 = \left[\frac{m'}{a'^2a}\frac{\partial}{\partial\alpha}(b_1\sin\phi+\ldots+ib_i\sin i\phi\ldots)\right.$$
$$\left.-\frac{m'}{a'a^2}(b_1\sin\phi+\ldots+ib_i\sin i\phi\ldots)-\sum_\mu m_\mu\left\{\frac{3\cos(\mu-\lambda)\,\pi/n}{8a^3\sin^2(\mu-\lambda)\,\pi/n}\right\}\right]\rho_\lambda$$
$$+\sum_\mu m_\mu\left[-\frac{\delta\cos(\mu-\lambda)\,\pi/n}{8a^3\sin^2(\mu-\lambda)\,\pi/n}+\frac{2}{a^3}\sin(\mu-\lambda)\,2\pi/n\right]\rho_\mu,$$

$$\sum_\mu\sigma_\mu\left(\frac{\partial^2\mathbf{F}}{r_\lambda\partial\theta_\mu\partial\theta_\lambda}\right)_0 = \left[-\frac{m'}{a'a}(b_1\cos\phi+\ldots+i^2b_i\cos i\phi\ldots)+\frac{m'}{a'^2}\cos\phi\right]\sigma_\lambda$$
$$+\sum_\mu m_\mu\left[\frac{\cos(\mu-\lambda)\,2\pi/n}{8a^2\sin^3(\mu-\lambda)\,\pi/n}-\frac{3\sin^2(\mu-\lambda)\,2\pi/n}{32a^2\sin^5(\mu-\lambda)\,\pi/n}\right.$$
$$\left.-\frac{1}{a^2}\cos(\mu-\lambda)\,2\pi/n\right](\sigma_\mu-\sigma_\lambda).$$

In the summations of the right-hand members, μ takes all integral values from 1 to n, except $\mu=\lambda$.

Next assume that all the small particles forming the ring are equal to one another. That is $m_\lambda = m$.

Further, let $\rho_{\lambda+1}=\beta\rho_\lambda$, for all values of λ.

Then

$$\rho_\lambda=\rho_{\lambda+n}=\beta^n\rho_\lambda,$$

whence

$$\beta^n=1\,;$$

or

$$\beta=\cos\frac{2s\pi}{n}+\iota\sin\frac{2s\pi}{n},\text{ where }\iota=\surd(-1),$$

and s takes all integral values from 0 to $n-1$.

The quantities appearing under the signs of summation are then :

$$\sum_\mu\frac{m}{a^2}\cos(\mu-\lambda)\,2\pi/n=-\frac{m}{a^2}\,;$$

$$\sum_\mu\frac{m}{a^2}\sin(\mu-\lambda)\,2\pi/n=0\,;$$

$$\sum_\mu\frac{m}{4a^2\sin(\mu-\lambda)\,\pi/n}=\frac{m}{a^2}\mathrm{K}\,;$$

$$\sum_\mu \frac{m}{8a^3}\left[\frac{\cos(\mu-\lambda)\,2\pi/n\,\{\cos s(\mu-\lambda)\,2\pi/n+\iota\sin s(\mu-\lambda)\,2\pi/n\}-1}{\sin^3(\mu-\lambda)\,\pi/n}\right]$$
$$+\sum_\mu \frac{3m}{8a^3}\left[\frac{\cos s(\mu-\lambda)2\pi/n+\iota\sin s(\mu-\lambda)2\pi/n+1}{\sin(\mu-\lambda)\,\pi/n}\right]$$
$$=\sum_\mu \frac{m}{a^3}\left\{\frac{1}{2}\frac{\cos^2 s(\mu-\lambda)\,\pi/n}{\sin(\mu-\lambda)\,\pi/n}-\frac{1}{4}\frac{\sin^2 s(\mu-\lambda)\,\pi/n\cos^2(\mu-\lambda)\,\pi/n}{\sin^3(\mu-\lambda)\,\pi/n}\right\}=-\frac{m}{a^3}\mathrm{L}_s\,;$$

$$\sum_\mu \frac{m}{8a^2}\frac{\cos(\mu-\lambda)\,\pi/n}{\sin^2(\mu-\lambda)\,\pi/n}\{\cos s(\mu-\lambda)\,\pi/n+\iota\sin s(\mu-\lambda)\,\pi/n-1\}$$
$$=\iota\frac{m}{8a^2}\sum_\mu\frac{\cos(\mu-\lambda)\,\pi/n\sin s(\mu-\lambda)\,\pi/n}{\sin^2(\mu-\lambda)\,\pi/n}=\iota\frac{m}{a^2}\mathrm{M}_s\,;$$

$$\sum_\mu \frac{m}{8a^3}\frac{\cos(\mu-\lambda)\,\pi/n}{\sin^2(\mu-\lambda)\,\pi/n}\{\cos s(\mu-\lambda)\,\pi/n+\iota\sin s(\mu-\lambda)\,\pi/n+3\}=\iota\frac{m}{a^3}\mathrm{M}_s\,;$$

$$\sum_\mu \frac{m}{8a^2}\left\{\frac{\cos(\mu-\lambda)\,2\pi/n}{\sin^3(\mu-\lambda)\,\pi/n}-\frac{3}{4}\frac{\sin^2(\mu-\lambda)\,2\pi/n}{\sin^5(\mu-\lambda)\,\pi/n}\right\}\{\cos s(\mu-\lambda)2\pi/n+\iota\sin s(\mu-\lambda)\,2\pi/n-1\}$$
$$=\frac{m}{a^2}\sum_\mu\left\{\frac{1}{2}\frac{\sin^2 s(\mu-\lambda)\,\pi/n\,.\cos^2(\mu-\lambda)\,\pi/n}{\sin^3(\mu-\lambda)\,\pi/n}+\frac{1}{4}\frac{\sin^2 s(\mu-\lambda)\,\pi/n}{\sin(\mu-\lambda)\,\pi/n}\right\}=\frac{m}{a^2}\mathrm{N}_s.$$

The quantities K, L_s, M_s, N_s can readily be found by direct summation when n, the number of particles, and s are known.

In re-writing the differential equations (2), we may now omit the suffixes of ρ and σ. Change the independent variable from t to $\phi=(\omega'-\omega)\,t+\epsilon'-\epsilon-\lambda\,.\,2\pi/n$. Also put $m/\mathrm{M}=\nu$, $m'/\mathrm{M}=\nu'$, $\omega'/\omega=\kappa'$, $(\kappa'-1)^{-1}=\kappa$, and, to secure homogeneity, replace ρ by $a\rho$. Let us further assume that $\omega^2a^3=\mathrm{M}$, and $\omega'^2a'^3=\mathrm{M}$ (the latter holds very approximately when m' describes a circle), so that we have $a/a'=(\omega'/\omega)^{2/3}$.

The differential equations then become

$$\left.\begin{aligned}
\frac{d^2\rho}{d\phi^2}-2\kappa\frac{d\sigma}{d\phi}&=\kappa^2\kappa'^{4/3}\nu'\frac{\partial}{\partial\alpha}(\tfrac{1}{2}b_0+\ldots+b_i\cos i\phi+\ldots)-\mathrm{K}\kappa^2\nu-\kappa^2\kappa'^{4/3}\nu'\cos\phi\\
&\quad+\left[3\kappa^2+\kappa^2\kappa'^2\nu'\frac{\partial^2}{\partial\alpha^2}(\tfrac{1}{2}b_0+\ldots+b_i\cos i\phi+\ldots)-\kappa^2\nu\mathrm{L}_s\right]\rho\\
&\quad+\left[\kappa^2\kappa'^{4/3}\nu'\frac{\partial}{\partial\alpha}(b_1\sin\phi+\ldots+ib_i\sin i\phi+\ldots-\sin\phi)+\kappa^2\nu\iota\mathrm{M}_s\right]\sigma,\\
&\text{and}\\
\frac{d^2\sigma}{d^2\phi}+2\kappa\frac{d\rho}{d\phi}&=\nu'\kappa^2\kappa'^{2/3}(b_1\sin\phi+\ldots+ib_i\sin i\phi+\ldots)-\nu'\kappa^2\kappa'^{4/3}\sin\phi\\
&\quad+\left[\nu'\kappa^2\kappa'^{4/3}\left(\ldots+i\frac{\partial b_i}{\partial\alpha}\sin i\phi+\ldots\right)-\nu'\kappa^2\kappa'^{2/3}(\ldots+ib_i\sin i\phi+\ldots)\right.\\
&\qquad\left.-\nu\kappa^2\iota\mathrm{M}_s\right]\rho\\
&\quad+\left[-\nu'\kappa^2\kappa'^{2/3}(\ldots+i^2b_i\cos i\phi+\ldots)+\nu'\kappa^2\kappa'^2\cos\phi+\nu\kappa^2\mathrm{N}_s\right]\sigma\,.
\end{aligned}\right\}\quad(3)$$

The equations (2) may be replaced, under the suppositions made, by equations (3). Equations (2) form a system of n pairs of linear equations of the second order. The complete integral will therefore involve $4n$ arbitrary constants. The system of equations (3) will give the same result, for the solution of (3) will be a function of s involving four arbitrary constants. By giving s its n values, 0, 1, 2, ..., $(n-1)$, we arrive at the complete integral involving $4n$ arbitrary constants.

Now it has been shown by TISSERAND* that for large values of n, whatever the value of s may be, the limiting value of L_s is $0{\cdot}0194n^3$, $N_s = 2L_s$ and $M_s = 0$.

These values largely simplify the discussion of the stability of the system.

Lastly the equations (3) may be written, for convenience, in the form

$$\left.\begin{aligned}
&\rho'' - 2\kappa\sigma' + (\Theta_{1,0} + \Theta_{1,1}\cos\phi + \ldots + \Theta_{1,r}\cos r\phi \ldots)\rho \\
&\qquad + (\Theta_{2,1}\sin\phi + \Theta_{2,2}\sin 2\phi + \ldots + \Theta_{2,r}\sin r\phi + \ldots)\sigma \\
&\qquad = \Theta_{3,0} + \Theta_{3,1}\cos\phi + \ldots + \Theta_{3,r}\cos r\phi + \ldots \\
&\sigma'' + 2\kappa\rho' + (\Theta_{4,1}\sin\phi + \Theta_{4,2}\sin 2\phi + \ldots + \Theta_{4,r}\sin r\phi + \ldots)\rho \\
&\qquad + (\Theta_{5,0} + \Theta_{5,1}\cos\phi + \ldots + \Theta_{5,r}\cos r\phi + \ldots)\sigma \\
&\qquad = \Theta_{6,1}\sin\phi + \ldots + \Theta_{6,r}\sin r\phi + \ldots
\end{aligned}\right\} \quad \ldots \quad (4)$$

The values of the quantities Θ are:

$$\left.\begin{aligned}
\Theta_{1,0} &= -3\kappa^2 - \tfrac{1}{2}\nu'\kappa^2\kappa'^2\frac{\partial^2 b_0}{\partial\alpha^2} + \nu\kappa^2 L_s && ; \\
\Theta_{1,r} &= -\nu'\kappa^2\kappa'^2\frac{\partial^2 b_r}{\partial\alpha^2} && ; \quad (r \neq 0) \\
\Theta_{2,r} &= -\nu'\kappa^2\kappa'^{4/3} r\frac{\partial b_r}{\partial\alpha} && ; \\
\Theta_{3,0} &= \tfrac{1}{2}\nu'\kappa^2\kappa'^{4/3}\frac{\partial b_0}{\partial\alpha} - \nu\kappa^2 K && ; \\
\Theta_{3,1} &= \nu'\kappa^2\kappa'^{4/3}\frac{\partial b_1}{\partial\alpha} - \nu'\kappa^2\kappa'^{4/3} && ; \\
\Theta_{3,r} &= \nu'\kappa^2\kappa'^{4/3}\frac{\partial b_r}{\partial\alpha} && ; \quad (r \neq 0,\ 1) \\
\Theta_{4,r} &= -\nu'\kappa^2\kappa'^{4/3} r\frac{\partial b_r}{b\alpha} + \nu'\kappa^2\kappa'^{2/3} r b_r && ; \\
\Theta_{5,0} &= -\nu\kappa^2 N_s && ; \\
\Theta_{5,r} &= \nu'\kappa^2\kappa'^{2/3} r^2 b_r && ; \quad (r \neq 0) \\
\Theta_{6,r} &= \nu'\kappa^2\kappa'^{2/3} r b_r && ; \quad (r \neq 1) \\
\Theta_{6,1} &= \nu'\kappa^2\kappa'^{2/3} b_1 - \nu'\kappa^2\kappa'^{4/3} && .
\end{aligned}\right\} \quad \ldots \quad (5)$$

* 'Méc. Céleste,' vol. ii., p. 184.

The best methods of determining the values of b_r, and its first and second derivatives for known values of α, or a/a', are given by TISSERAND.* The complete evaluation of a number of these quantities for various ratios, applicable to the solar system, is given by PONTÉCOULANT.† For the purpose of estimating the order of the numerical values of the quantities $\Theta_{r,s}$, we may take the highest ratio α likely to occur as that of the outer edge of the ring to the mean distance of Mimas. This ratio is 0·7461 (see Appendix for data). PONTÉCOULANT gives the values for $\alpha = 0{\cdot}72333$, which we may use to avoid laborious calculation. If we take $\nu' = 7 \,.\, 10^{-8}$, the value for Mimas, we find

$$\Theta_{1,0} = -20{\cdot}2590 + 6{\cdot}7530\nu L_s;$$

$$\Theta_{1,1} = -1{\cdot}35 \,.\, 10^{-6}, \qquad \Theta_{1,2} = -1{\cdot}53 \,.\, 10^{-6}, \qquad \Theta_{1,3} = -1{\cdot}63 \,.\, 10^{-6} \ldots;$$

$$\Theta_{2,1} = -4{\cdot}07 \,.\, 10^{-7}, \qquad \Theta_{2,2} = -7{\cdot}39 \,.\, 10^{-7}, \qquad \Theta_{2,3} = -9{\cdot}34 \,.\, 10^{-6} \ldots;$$

$$\Theta_{3,0} = -1{\cdot}46 \,.\, 10^{-7} - 1{\cdot}79 \,.\, 10^{-7}K, \qquad \Theta_{3,1} = 4{\cdot}07 \,.\, 10^{-7}, \qquad \Theta_{3,2} = 3{\cdot}69 \,.\, 10^{-7} \ldots$$

$$\Theta_{5,0} = -13{\cdot}5060\nu L_s.$$

It is clear that, compared with $\Theta_{1,0}$, all products and squares of the remaining Θ's may be neglected.

§ 3. *Solution of the Equations.*

(a) The complementary function.

The equations

$$\left.\begin{aligned} \rho'' - 2\kappa\sigma' + \rho\Sigma\Theta_{1,r}\cos r\phi + \sigma\Sigma\Theta_{2,r}\sin r\phi &= 0, \\ \sigma'' + 2\kappa\rho' + \rho\Sigma\Theta_{4,r}\sin r\phi + \sigma\Sigma\Theta_{5,r}\cos r\phi &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

belong to the class of homogeneous linear differential equations with periodic coefficients. The integral is known to be the sum of the forms $e^{c\phi}f(\phi)$, where $f(\phi)$ is a periodic function of ϕ with the same period as the coefficients in the equations (6). Equations of this form in one dependent variable have been discussed by WHITTAKER,‡ YOUNG,§ INCE,‖ and BAKER.¶ The present solution is a simple extension of the work of these writers.

Let

$$\rho = e^{c\phi}A,$$
$$\sigma = e^{c\phi}X,$$

* 'Méc. Céleste,' vol. i., p. 270, *et seq.*

† 'Système du Monde,' vol. 3, pp. 353–376.

‡ 'Proc Inter. Congress Math.' vol. 1, 1912; 'Proc. Edin. Math. Soc.' xxxii., p. 76.

§ 'Proc. Edin. Math. Soc.' xxxii., p. 81.

‖ 'Monthly Notices R.A.S.' lxxv., 5, p. 436.

¶ H. F. BAKER, 'Phil. Trans.' A., vol. 216, p. 129.

where A and X are, as has been said, purely periodic functions of period 2π. On substituting in equations (6) we find

$$\left.\begin{array}{l} c^2\mathrm{A}+2c\mathrm{A}'+\mathrm{A}''-2\kappa\,(c\mathrm{X}+\mathrm{X}')+\mathrm{A}\Sigma\Theta_{1,r}\cos r\phi+\mathrm{X}\Sigma\Theta_{2,r}\sin r\phi = 0 \\ c^2\mathrm{X}+2c\mathrm{X}'+\mathrm{X}''+2\kappa\,(c\mathrm{A}+\mathrm{A}')+\mathrm{A}\Sigma\Theta_{4,r}\sin r\phi+\mathrm{X}\Sigma\Theta_{5,r}\sin r\phi = 0 \end{array}\right\} \quad . \quad . \quad (7)$$

Let us now assume that A and X can be represented in the most general way by a series of terms in Θ with suitable coefficients, the coefficients being periodic functions of ϕ with period 2π. That is, let

$$\mathrm{A} = \mathrm{A}_0 \sin(n\phi-\tau)+\Sigma\Sigma\mathrm{A}_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma\mathrm{B}_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots,$$
$$\mathrm{X} = \mathrm{X}_0 \cos(n\phi-\tau)+\Sigma\Sigma\mathrm{X}_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma\mathrm{Y}_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots.$$

In these expressions A_0 and X_0 will be arbitrary constants, n is an arbitrary integer* and τ a parameter which will be defined presently.

We shall assume that the index c is of the form

$$c = \Sigma\Sigma c_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma d_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots.$$

Then, if we substitute these values in equations (7) and equate to zero those terms which do not involve any Θ except $\Theta_{1,0}$ and $\Theta_{5,0}$, which are large compared with the others, we find

$$\left.\begin{array}{l} \{(\Theta_{1,0}-n^2)\,\mathrm{A}_0+2\kappa n\mathrm{X}_0\}\sin(n\phi-\tau) = 0, \\ \{2\kappa n\mathrm{A}_0+(\Theta_{5,0}-n^2)\,\mathrm{X}_0\}\cos(n\phi-\tau) = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

On eliminating A_0 and X_0 we find

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2)-4\kappa^2n^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In general, the given values of $\Theta_{1,0}$ and $\Theta_{5,0}$ will not satisfy the identity (9) for any integral value of n. Let us replace $\Theta_{1,0}$ by $a_{1,0}$, where $a_{1,0}$ is a quantity which satisfies the relation

$$(a_{1,0}-n^2)(\Theta_{5,0}-n^2)-4\kappa^2n^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For some suitable value of n, it will usually be found that $a_{1,0}$ approximates closely to $\Theta_{1,0}$.

Following the method of WHITTAKER previously referred to, let us now assume that

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2)-(a_{1,0}-n^2)(\Theta_{5,0}-n^2) = \Sigma\Sigma u_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma v_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots,$$

or

$$\Theta_{1,0} = n^2+\frac{4\kappa^2n^2}{\Theta_{5,0}-n^2}+\Sigma\Sigma a_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma b_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots \quad . \quad . \quad . \quad . \quad (11)$$

* The use of n is to be distinguished from a former use where it referred to the number of particles in the ring.

We now substitute the assumed values for A, X, c, $\Theta_{1,0}$ in equations (7) and equate to zero the coefficients of each term in $\Theta_{r,s}$, $\Theta_{r,s}$, $\Theta_{p,q}$, &c. It will be found that the relations (8) are satisfied identically. Two conditions further must be imposed in order that all the unknown coefficients may be determined. These are:

(i) The term $\cos(n\phi-\tau)$ must not appear in the series for A;

(ii) The solutions for A and X must be purely periodic with period 2π.

The condition (i) amounts to a definition of τ, and condition (ii) secures that no part of the exponent shall appear in the periodic series. Further, these conditions determine uniquely the undetermined coefficients in the series for $\Theta_{1,0}$ and c. The work from this point is purely mechanical though long. The following sample sufficiently indicates its character.

On equating to zero the terms involving $\Theta_{1,r}$ we find

$$\left.\begin{aligned} &2c_{1,r}n\mathrm{A}_0\cos(n\phi-\tau)+\mathrm{A}''_{1,r}-2\kappa c_{1,r}\mathrm{X}_0\cos(n\phi-\tau)\\ &\qquad -2\kappa\mathrm{X}'_{1,r}+a_{1,0}\mathrm{A}_{1,r}+a_{1,r}\mathrm{A}_0\sin(n\phi-\tau)+\mathrm{A}_0\cos r\phi\sin(n\phi-\tau)=0\\ &-2c_{1,r}n\mathrm{X}_0\sin(n\phi-\tau)+\mathrm{X}''_{1,r}+2\kappa c_{1,r}\mathrm{A}_0\sin(n\phi-\tau)+2\kappa\mathrm{A}'_{1,r}+\Theta_{5,0}\mathrm{X}_{1,r}=0\end{aligned}\right\}\quad(12)$$

In the case when r is not $2n$ or n, it is clear that

$$c_{1,r}=0 \qquad \text{and} \qquad a_{1,r}=0.$$

Equation (12) then reduces to

$$\left.\begin{aligned} &\mathrm{A}''_{1,r}-2\kappa\mathrm{X}'_{1,r}+a_{1,0}\mathrm{A}_{1,r}+\tfrac{1}{2}\mathrm{A}_0[\sin\{(n+r)\phi-\tau\}+\sin\{(n-r)\phi-\tau\}]=0,\\ &\mathrm{X}''_{1,r}+2\kappa\mathrm{A}'_{1,r}+\Theta_{5,0}\mathrm{X}_{1,r}=0.\end{aligned}\right\}\quad . \quad (13)$$

Solving in the usual way we find

$$\mathrm{A}_{1,r}=-\frac{\mathrm{A}_0n^2\{(n+r)^2-\Theta_{5,0}\}\sin\{(n+r)\phi-\tau\}}{2r(2n+r)(a_{1,0}\Theta_{5,0}-n^2(n+r)^2)}+\frac{\mathrm{A}_0n^2\{(n-r)^2-\Theta_{5,0}\}\sin\{(n-r)\phi-\tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}},$$

$$\mathrm{X}_{1,r}=-\frac{\mathrm{A}_0n^2\kappa(n+r)\cos\{(n+r)\phi-\tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0}-n^2(n+r)^2\}}+\frac{\mathrm{A}_0n^2\kappa(n-r)\cos\{(n-r)\phi-\tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}}.$$

In the special case where $r=n$, we have

$$c_{1,n}=0, \qquad a_{1,n}=0,$$

and

$$\left.\begin{aligned} &\mathrm{A}''_{1,n}-2\kappa\mathrm{X}'_{1,n}+a_{1,0}\mathrm{A}_{1,n}+\tfrac{1}{2}\mathrm{A}_0\{\sin(2n\phi-\tau)-\sin\tau\}=0,\\ &\mathrm{X}''_{1,n}+2\kappa\mathrm{A}'_{1,n}+\Theta_{5,0}\mathrm{X}_{1,n}=0.\end{aligned}\right\}\quad . \quad . \quad . \quad (14)$$

From which

$$\mathrm{A}_{1,n}=\frac{\mathrm{A}_0(4n^2-\Theta_{5,0})\sin(2n\phi-\tau)}{6(4n^4-a_{1,0}\Theta_{5,0})}+\frac{\mathrm{A}_0}{2a_{1,0}}\sin\tau,$$

and

$$\mathrm{X}_{1,n}=\tfrac{2}{3}\frac{\kappa n\mathrm{A}_0\cos(2n\phi-\tau)}{4n^4-a_{1,0}\Theta_{5,0}}.$$

Again, in the special case where $r = 2n$, we find in place of equations (12), the following

$$\left.\begin{aligned} &2c_{1,2n}nA_0\cos(n\phi-\tau)+A''_{1,2n}-2\kappa c_{1,2n}X_0\cos(n\phi-\tau)-2\kappa X'_{1,2n} \\ &\quad +a_{1,0}A_{1,2n}+a_{1,2n}A_0\sin(n\phi-\tau)+\tfrac{1}{2}A_0\{\sin(3n\phi-\tau)-\sin(n\phi-\tau)\cos 2\tau \\ &\qquad\qquad -\cos(n\phi-\tau)\sin 2\tau\} = 0, \\ &\quad -2c_{1,2n}nX_0\sin(n\phi-\tau)+X''_{1,2n}+2\kappa c_{1,2n}A_0\sin(n\phi-\tau)+2\kappa A'_{1,2n}+\Theta_{5,0}X_{1,2n} = 0. \end{aligned}\right\} \quad (15)$$

In order to avoid the explicit appearance of ϕ, we must have

$$a_{1,2n} = \tfrac{1}{2}\cos 2\tau.$$

Since we have already stipulated that A must not contain any term in $\cos(n\phi-\tau)$, $c_{1,2n}$ must be so chosen as to make quantities involving $\cos(n\phi-\tau)$ annul. Hence we must have

$$\left.\begin{aligned} \{2c_{1,2n}nA_0-2\kappa c_{1,2n}X_0-\tfrac{1}{2}A_0\sin 2\tau\}\cos(n\phi-\tau)-2\kappa' X'_{1,2n} &= 0, \\ \{-2c_{1,2n}nX_0+2\kappa c_{1,2n}A_0\}\sin(n\phi-\tau)+X''_{1,2n}+\Theta_{5,0}X_{1,2n} &= 0. \end{aligned}\right\} \quad . \quad . \quad (16)$$

Whence

$$c_{1,2n} = \frac{n(\Theta_{5,0}-n^2)\sin 2\tau}{4(a_{1,0}\Theta_{5,0}-n^4)},$$

$$X_{1,2n} = \frac{1}{8}\frac{(n^2-a_{1,0})(\Theta_{5,0}+n^2)\sin 2\tau A_0\sin(n\phi-\tau)}{\kappa n(a_{1,0}\Theta_{5,0}-n^4)}.$$

To the value for $X_{1,2n}$ must be added the further particular solution arising from the term $\frac{1}{2}A_0\sin(3n\phi-\tau)$ in (15). It is

$$A_{1,2n} = \frac{(\Theta_{5,0}-9n^2)A_0\sin(3n\phi-\tau)}{16(a_{1,0}\Theta_{5,0}-9n^4)},$$

$$X_{1,2n} = \frac{3n\kappa A_0\cos(3n\phi-\tau)}{8(a_{1,0}\Theta_{5,0}-9n^4)}.$$

Proceeding in this way, we have the following results:—

Terms not involving argument Θ:

In A

$$A_0\sin(n\phi-\tau).$$

In X

$$X_0\cos(n\phi-\tau).$$

In c

None.

Also

$$2\kappa nX_0 = -(a_{1,0}-n^2)A_0,$$

when

$$a_{1,0} = n^2+4\kappa^2n^2/(\Theta_{5,0}-n^2).$$

Terms involving argument $\Theta_{1,r}$, *where* r *is not* n *nor* $2n$:

$$a_{1,r} = 0, \qquad c_{1,r} = 0,$$

$$A_{1,r} = -\frac{A_0 n^2 \{(n+r)^2 - \Theta_{5,0}\} \sin\{(n+r)\phi - \tau\}}{2r(2n+r)(a_{1,0}\Theta_{5,0} - n^2(n+r)^2)} + \frac{A_0 n^2 \{(n-r)^2 - \Theta_{5,0}\} \sin\{(n-r)\phi - \tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}},$$

$$X_{1,r} + -\frac{A_0 n^2 \kappa (n+r) \cos\{(n+r)\phi - \tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} + \frac{A_0 n^2 \kappa (n-r) \cos\{(n-r)\phi - \tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{1,n}$:

$$a_{1,n} = 0, \qquad c_{1,n} = 0,$$

$$A_{1,n} = \frac{A_0(4n^2 - \Theta_{5,0}) \sin(2n\phi - \tau)}{6(4n^4 - a_{1,0}\Theta_{5,0})} + \frac{A_0 \sin\tau}{2a_{1,0}},$$

$$X_{1,n} = \frac{2\kappa n A_0 \cos(2n\phi - \tau)}{3(4n^4 - a_{1,0}\Theta_{5,0})}.$$

Terms involving argument $\Theta_{1,2n}$:

$$a_{1,2n} = \tfrac{1}{2}\cos 2\tau, \qquad c_{1,2n} = \frac{n(\Theta_{5,0} - n^2)\sin 2\tau}{4(a_{1,0}\Theta_{5,0} - n^4)},$$

$$A_{1,2n} = \frac{(\Theta_{5,0} - 9n^2) A_0 \sin(3n\phi - \tau)}{16(a_{1,0}\Theta_{5,0} - 9n^4)},$$

$$X_{1,2n} = -\frac{3n\kappa A_0 \cos(3n\phi - \tau)}{8(a_{1,0}\Theta_{5,0} - 9n^4)} + \frac{(n^2 - a_{1,0})(\Theta_{5,0} - n^2)\sin 2\tau A_0 \sin(n\phi - \tau)}{8\kappa n(a_{1,0}\Theta_{5,0} - 9n^4)}.$$

Terms involving argument $\Theta_{2,r}$, *where* r *is not* n *nor* $2n$:

$$a_{2,r} = 0, \qquad c_{2,r} = 0,$$

$$A_{2,r} = \frac{n^2\{\Theta_{5,0} - (n+r)^2\} X_0 \sin\{(n+r)\phi - \tau\}}{4r(n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} + \frac{n^2\{\Theta_{5,0} - (n-r)^2\} X_0 \sin\{(n-r)\phi - \tau\}}{4r(n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}},$$

$$X_{2,r} = \frac{X_0 n^2 \kappa (n+r) \cos\{(n+r)\phi - \tau\}}{2r(n+r)\{a_{1,0}\Theta_{5,0} - n^2(n+r)^2\}} + \frac{X_0 n^2 \kappa (n-r) \cos\{(n-r)\phi - \tau\}}{2r(n-r)\{a_{1,0}\Theta_{5,0} - n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{2,n}$:

$$a_{2,n} = 0, \qquad c_{2,n} = 0,$$

$$A_{2,n} = \frac{X_0(\Theta_{5,0} - 4n^2)\sin(2n\phi - \tau)}{6(a_{1,0}\Theta_{5,0} - 4n^4)} - \frac{X_0 \sin 2\tau}{2a_{1,0}},$$

$$X_{2,n} = -\frac{2X_0 \kappa n \cos(2n\phi - \tau)}{3(a_{1,0}\Theta_{5,0} - 4n^4)}.$$

Terms involving argument $\Theta_{2,2n}$:

$$a_{2,2n} = -\frac{\kappa n \cos 2\tau}{\Theta_{5,0}-n^2}, \qquad c_{2,2n} = \frac{\kappa n^2 \sin 2\tau}{2(n^4-a_{1,0}\Theta_{5,0})},$$

$$A_{2,2n} = \frac{(\Theta_{5,0}-9n^2)X_0 \sin(3n\phi-\tau)}{9n^4-a_{1,0}\Theta_{5,0}},$$

$$X_{2,2n} = \frac{3\kappa n X_0 \cos(3n\phi-\tau)}{8(9n^4-a_{1,0}\Theta_{5,0})} + \frac{(\Theta_{5,0}+n^2)\kappa n X_0 \sin 2\tau \sin(n\phi-\tau)}{2(\Theta_{5,0}-n^2)(n^4-a_{1,0}\Theta_{5,0})}.$$

Terms involving argument $\Theta_{4,r}$, *where* r *is not* n *nor* $2n$:

$$c_{4,r} = 0, \qquad a_{4,r} = 0,$$

$$A_{4,r} = \frac{A_0 n^2\kappa(n+r)\sin\{(n+r)\phi-\tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0}-n^2(n+r)^2\}} + \frac{A_0 n^2\kappa(n-r)\sin\{(n-r)\phi-\tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}},$$

$$X_{4,r} = \frac{A_0 n^2\{(n+r)^2-a_{1,0}\}\cos\{(n+r)\phi-\tau\}}{2r(2n+r)\{a_{1,0}\Theta_{5,0}-n^2(n+r)^2\}} + \frac{A_0 n^2\{(n-r)^2-a_{1,0}\}\cos\{(n-r)\phi-\tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{4,n}$:

$$c_{4,n} = 0, \qquad a_{4,n} = 0,$$

$$A_{4,n} = \frac{2n\kappa A_0 \sin(2n\phi-\tau)}{3(a_{1,0}\Theta_{5,0}-4n^4)},$$

$$X_{4,n} = \frac{A_0(4n^2-a_{1,0})\cos(2n\phi-\tau)}{6(a_{1,0}\Theta_{5,0}-4n^4)} - \frac{A_0\cos\tau}{2\Theta_{5,0}}.$$

Terms involving argument $\Theta_{4,2n}$:

$$a_{4,2n} = \frac{n\kappa\cos 2\tau}{\Theta_{5,0}-n^2}, \qquad c_{4,2n} = \frac{n^2\kappa\sin 2\tau}{2(a_{1,0}\Theta_{5,0}-n^4)},$$

$$A_{4,2n} = \frac{3\kappa n A_0 \sin(3n\phi-\tau)}{8(a_{1,0}\Theta_{5,0}-9n^4)},$$

$$X_{4,2n} = \frac{(9n^2-a_{1,0})A_0\cos(3n\phi-\tau)}{16(a_{1,0}\Theta_{5,0}-9n^4)} + \frac{A_0\cos 2\tau\cos(n\phi-\tau)}{2(\Theta_{5,0}-n^2)} + \frac{(a_{1,0}+n^2)\sin 2\tau A_0\sin(n\phi-\tau)}{4(a_{1,0}\Theta_{5,0}-n^4)}$$

Terms involving argument $\Theta_{5,r}$, *when* r *is not* n *nor* $2n$:

$$c_{5,r} = 0, \qquad a_{5,r} = 0,$$

$$A_{5,r} = \frac{X_0 n^2\kappa(n+r)\sin\{(n+r)\phi-\tau\}}{r(2n+r)\{a_{1,0}\Theta_{5,0}-n^2(n+r)^2\}} - \frac{X_0 n^2\kappa(n-r)\sin\{(n-r)\phi-\tau\}}{r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}},$$

$$X_{5,r} = \frac{X_0 n^2\{(n+r)^2-a_{1,0}\}\cos\{(n+r)\phi-\tau\}}{2r(2n+r)\{a_{1,0}\Theta_{5,0}-n^2(n+r)^2\}} - \frac{X_0 n^2\{(n-r)^2-a_{1,0}\}\cos\{(n-r)\phi-\tau\}}{2r(2n-r)\{a_{1,0}\Theta_{5,0}-n^2(n-r)^2\}}.$$

Terms involving argument $\Theta_{5,n}$:

$$a_{5,n} = 0, \qquad c_{5,n} = 0,$$

$$A_{5,n} = \frac{2X_0\kappa 2n \sin(2n\phi-\tau)}{3(a_{1,0}\Theta_{5,0}-4n^4)} - \frac{X_0\cos\tau}{2\Theta_{5,0}},$$

$$X_{5,n} = \frac{X_0(4n^2-a_{1,0})\cos(2n\phi-\tau)}{6(a_{1,0}\Theta_{5,0}-4n^4)}.$$

Terms involving argument $\Theta_{5,2n}$:

$$a_{5,2n} = -\frac{(a_{1,0}-n^2)\cos 2\tau}{2(\Theta_{5,0}-n^2)}, \qquad c_{5,2n} - \frac{n(a_{1,0}-n^2)\sin 2\tau}{4(a_{1,0}\Theta_{5,0}-n^4)},$$

$$A_{5,2n} = \frac{3n\kappa X_0 \sin(3n\phi-\tau)}{8(9n^4-a_{1,0}\Theta_{5,0})},$$

$$X_{5,2n} = \frac{9(n^2-a_{1,0})X_0\cos(3n\phi-\tau)}{16(9n^4-a_{1,0}\Theta_{5,0})} + \frac{(a_{1,0}-n^2)\cos 2\tau A_0\cos(n\phi-\tau)}{4\kappa n(\Theta_{5,0}-n^2)}$$
$$- \frac{(a_{1,0}+n^2)(a_{1,0}-n^2)\sin 2\tau A_0 \sin(n\phi-\tau)}{8\kappa n(a_{1,0}\Theta_{5,0}-n^4)}.$$

Terms involving powers products of the Θ's follow in similar fashion.

If we summarize the parts specially required, we find

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2+\tfrac{1}{2}(\Theta_{5,0}-n^2)\cos 2\tau\Theta_{1,2n}-\kappa n\cos 2\tau\Theta_{2,2n}+\kappa n\cos 2\tau\Theta_{4,2n}$$
$$-\tfrac{1}{2}(a_{1,0}-n^2)\cos 2\tau\Theta_{5,2n}+\dots; \quad . \quad . \quad (17)$$

and

$$2c(a_{1,0}\Theta_{5,0}-n^4) = \tfrac{1}{2}n(\Theta_{5,0}-n^2)\sin 2\tau\Theta_{1,2n}-\kappa n^2\sin 2\tau\Theta_{2,2n}+\kappa n^2\sin 2\tau\Theta_{4,2n}$$
$$-\tfrac{1}{2}n(a_{1,0}-n^2)\sin 2\tau\Theta_{5,2n}+\dots; \quad . \quad . \quad (18)$$

where, as already stated,

$$(a_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2.$$

It is necessary to examine the expressions just obtained in order to see whether the complete integral of equations (6) has been found.

The integer n is determined so as most nearly to satisfy the relation

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2,$$

when

$$\Theta_{1,0},\ \Theta_{5,0} \text{ and } \kappa \text{ are known.}$$

The negative value of n will also satisfy this relation.

On solving equation (17), for each value of n there will be, in general, two values of 2τ, equal and opposite in sign. So that altogether there are four distinct values of 2τ obtainable. Each of these with the corresponding value of n will give a different value of c on substituting in equation (18), and different values for A and X. Hence there are four distinct solutions and these when multiplied by arbitrary constants will give the complete primitive of equations (6).

(*b*) The Particular Integral.

We have now to determine the particular integral of equations (4). We shall assume only one general term on the right-hand side and take the complete solution as the sum of a series of the corresponding solutions. The equations may therefore be written

$$\left.\begin{aligned}\rho''-2\kappa\sigma'+\rho\Sigma\Theta_{1,r}\cos r\phi+\sigma\Sigma\Theta_{2,r}\sin r\phi &= \tfrac{1}{2}\Theta_{3,m}e^{+\iota m\phi},\\ \sigma''+2\kappa\rho'+\rho\Sigma\Theta_{4,r}\sin r\phi+\sigma\Sigma\Theta_{5,r}\cos r\phi &= 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

Assume

$$\rho = e^{\iota m\phi}\mathrm{A},^*$$

and

$$\sigma = e^{\iota m\phi}\mathrm{X},$$

where X and A as before are functions of ϕ. On substituting in equations (19) and reducing, we find

$$\left.\begin{aligned}-m^2\mathrm{A}+2\iota m\mathrm{A}'+\mathrm{A}''-2\kappa(\iota m\mathrm{X}+\mathrm{X}')+\mathrm{A}\Sigma\Theta_{2,r}\cos r\phi+\mathrm{X}\Sigma\Theta_{2,r}\sin r\phi &= \tfrac{1}{2}\Theta_{3,m},\\ -m^2\mathrm{X}+2\iota m\mathrm{X}'+\mathrm{X}''+2\kappa(\iota m\mathrm{A}+\mathrm{A}')+\mathrm{A}\Sigma\Theta_{4,r}\sin r\phi+\mathrm{X}\Sigma\Theta_{5,r}\cos r\phi &= 0.\end{aligned}\right\} . \quad (20)$$

As a solution we now take

$$\mathrm{A} = \mathrm{A}_0+\Sigma\Sigma\mathrm{A}_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma\mathrm{B}_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots,$$
$$\mathrm{X} = \mathrm{X}_0+\Sigma\Sigma\mathrm{X}_{r,s}\Theta_{r,s}+\Sigma\Sigma\Sigma\Sigma\mathrm{Y}_{r,s,p,q}\Theta_{r,s}\Theta_{p,q}+\ldots.$$

In these summations all the Θ's in the coefficients of ρ and σ are to be included except $\Theta_{1,0}$ and $\Theta_{5,0}$. A_0 and X_0 are constants, and the other coefficients functions of ϕ.

Now substitute these expansions for A and X in (20), and equate to zero the terms involving no Θ except $\Theta_{1,0}$ and $\Theta_{5,0}$. We then have

$$-m^2\mathrm{A}_0-2\kappa\iota m\mathrm{X}_0+\Theta_{1,0}\mathrm{A}_0 = \tfrac{1}{2}\Theta_{3,m},$$
$$-m^2\mathrm{X}_0+2\kappa\iota m\mathrm{A}_0+\Theta_{5,0}\mathrm{X}_0 = 0.$$

Whence

$$\mathrm{A}_0 = \tfrac{1}{2}\Theta_{3,m}(\Theta_{5,0}-m^2)\div\{(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2\},$$
$$\mathrm{X}_0 = -\kappa\iota m\Theta_{3m}\div\{(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2\}.$$

Next, taking the coefficient of $\Theta_{1,r}$, we have the equations

$$\left.\begin{aligned}-m^2\mathrm{A}_{1,r}+2\iota m\mathrm{A}'_{1,r}+\mathrm{A}''_{1,r}-2\kappa(\iota m\mathrm{X}_{1,r}+\mathrm{X}'_{1,r})+\Theta_{1,0}\mathrm{A}_{1,0}+\mathrm{A}_0\cos r\phi &= 0\\ -m^2\mathrm{X}_{1,r}+2\iota m\mathrm{X}'_{1,r}+\mathrm{X}''_{1,r}+2\kappa(\iota m\mathrm{A}_{1,r}+\mathrm{A}'_{1,r})+\Theta_{5,0}\mathrm{X}_{1,r} &= 0\end{aligned}\right\} . \quad (21)$$

* The use of m here to represent an integer is to be carefully distinguished from its previous use to represent mass.

We shall form the solution by taking only $e^{\iota r\phi}$ in the term $\cos r\phi$. Changing the sign of r will then give the other part. Assuming that $A_{1,r}$ and $X_{1,r}$ vary as $e^{\iota r\phi}$, we have

$$\left.\begin{aligned} A_{1,r}(-m^2-2mr-r^2+\Theta_{1,0})-2\kappa\iota(m+r)X_{1,r} &= -\tfrac{1}{2}A_0,\\ A_{1,r}(m+r)2\kappa\iota+X_{1,r}(-m^2-2mr-r^2+\Theta_{5,0}) &= 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (22)$$

From these

$$A_{1,r} = -\tfrac{1}{2}A_0\{\Theta_{5,0}-(m+r)^2\} \div [\{\Theta_{1,0}-(m+r)^2\}\{\Theta_{5,0}-(m+r)^2\}-4\kappa^2(m+r)^2],$$

and

$$X_{1,r} = +\tfrac{1}{2}A_0 \,.\, 2\kappa\iota(m+r) \div [\{\Theta_{1,0}-(m+r)^2\}\{\Theta_{5,0}-(m+r)^2\}-4\kappa^2(m+r)^2].$$

On determining the corresponding values for the term $e^{-\iota r\phi}$ and combining the two, we have

$$\left.\begin{aligned} A_{1,r} = &-\tfrac{1}{2}A_0e^{\iota r\phi}\{\Theta_{5,0}-(m+r)^2\} \div [\{\Theta_{1,0}-(m+r)^2\}\{\Theta_{5,0}-(m+r)^2\}-4\kappa^2(m+r)^2]\\ &-\tfrac{1}{2}A_0e^{-\iota r\phi}\{\Theta_{5,0}-(m-r)^2\} \div [\{\Theta_{1,0}-(m-r)^2\}\{\Theta_{5,0}-(m-r)^2\}-4\kappa^2(m-r)^2],\\ X_{1,r} = &\ A_0e^{\iota r\phi}\kappa\iota(m+r) \div [\{\Theta_{1,0}-(m+r)^2\}\{\Theta_{5,0}-(m+r)^2\}-4\kappa^2(m+r)^2]\\ &+A_0e^{-\iota r\phi}\kappa\iota(m-r) \div [\{\Theta_{1,0}-(m-r)^2\}\{\Theta_{5,0}-(m-r)^2\}-4\kappa^2(m-r)^2].\end{aligned}\right\} \quad (23)$$

Expression (23) shows that $A_{1,r}$ and $X_{1,r}$ are factored by A_0, which is a multiple of $\Theta_{3,m}$. Now the terms in the expansions of A and X that we are seeking are $A_{1,r}\Theta_{1,r}$ and $X_{1,r}\Theta_{1,r}$. Since both of these involve the product $\Theta_{3,m}\Theta_{1,r}$, it is clear that they may be neglected in comparison with the values of A_0 and X_0.

We have further to determine the parts of A and X arising from a term $\dfrac{1}{2\iota}\Theta_{6,m}e^{\iota m\phi}$ in the right-hand member of the second equations (4). These can be written down from the results already given, and are

$$X_0 = \frac{1}{2\iota}\Theta_{6,m}(\Theta_{5,0}-m^2) \div \{(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2\},$$

$$A_0 = \kappa m\Theta_{6,m} \div \{(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2\}.$$

Hence to the degree of accuracy we are using, we may summarise the results as:

$$\left.\begin{aligned} \rho &= \sum_m [\Theta_{3,m}(\Theta_{5,0}-m^2)\cos m\phi+2\kappa m\Theta_{6,m}\cos m\phi] \div [(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2],\\ \sigma &= \sum_m [2\kappa m\Theta_{3,m}\sin m\phi+\Theta_{6,m}(\Theta_{5,0}-m^2)\sin m\phi] \div [(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2].\end{aligned}\right\} . \quad (24)$$

Except when the denominators are small, it is seen that, owing to the very small factors $\Theta_{3,m}$ and $\Theta_{6,m}$, the values of ρ and σ derived from the above equations are very small.

§ 4. *Discussion of the Solutions of the Equations for the Case of Equal Particles.*

(i.) The complementary function.

Equations (17) and (18) which determine the value of the exponent c, may be re-written here,

$$\left.\begin{aligned}(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2+\{\tfrac{1}{2}(\Theta_{5,0}-n^2)\Theta_{1,2n}\\ -\kappa n\Theta_{2,2n}+\kappa n\Theta_{4,2n}-\tfrac{1}{2}(\alpha_{1,0}-n^2)\Theta_{5,2n}\}\cos 2\tau\end{aligned}\right\} \quad . \quad . \quad (25)$$

$$\left.\begin{aligned}2c(\alpha_{1,0}\Theta_{5,0}-n^4) = \{\tfrac{1}{2}n(\Theta_{5,0}-n^2)\Theta_{1,2n}-\kappa n^2\Theta_{2,2n}\\ +\kappa n^2\Theta_{4,2n}-\tfrac{1}{2}n(\alpha_{1,0}-n^2)\Theta_{5,2n}\}\sin 2\tau\end{aligned}\right\} \quad . \quad . \quad (26)$$

In these $\alpha_{1,0}$ is determined by the relation

$$(\alpha_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2.$$

It is noticeable that the coefficient of $\sin 2\tau$ in (26) is n times that of $\cos 2\tau$ in (25). Owing to the smallness of the quantities Θ (excepting $\Theta_{1,0}$ and $\Theta_{5,0}$), it is clear that the coefficients of $\cos 2\tau$ and $\sin 2\tau$ are both very small quantities. Now real values of c are only given by real values of τ, and conversely. Hence in order that (25) may give real values of τ it is necessary that the expression

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2)-4\kappa^2n^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

should be less than, or at most equal to, the coefficient of $\cos 2\tau$. That is, the real values of c will be in the vicinity of these values of κ that make (27) vanish. The actual limits of the zone in which real values of c are found will be given by

$$(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2) = 4\kappa^2n^2\pm\{\tfrac{1}{2}(\Theta_{5,0}-n^2)\Theta_{1,2n}\mp\kappa n\Theta_{2,2n}\pm\kappa n\Theta_{4,2n}-\tfrac{1}{2}(\alpha_{1,0}-n^2)\Theta_{5,2n}\} \quad . \quad (28)$$

There are four groups of signs possible in this expression, and there will result four values of κ. The outermost and innermost of these will define the zone in which some real value of c appears, and this zone will be the zone of instability. Owing, however, to the extreme smallness of the coefficient of $\sin 2\tau$ in (26), it is clear that c will be extremely small, in general; that is, the modulus of instability will be small and departure from the zone will be slow. In one case, however, c may be quite large. That is, when the coefficient of c, $\alpha_{1,0}\Theta_{5,0}-n^4$, is exceedingly small.

Each of the quantities Θ is a function of $\frac{a}{a'}$ or of κ. Further $\Theta_{1,0}$ and $\Theta_{5,0}$ involve both the mass of the particles and the number of them. Both of these are entirely unknown. All that can be said is that MAXWELL'S criterion,* that is,

$$\nu < \frac{2}{p^3},$$

* TISSERAND, 'Méc. Céleste,' vol. ii., p. 184.

where ν is the ratio of the mass of a particle to the mass of Saturn and p is the number of particles in the ring, must be fulfilled.

ν appears in the expressions for $\Theta_{1,0}$ and $\Theta_{5,0}$ in the form νL_s. It has been mentioned that $L_s < 0{\cdot}0194n^3$ for all values of s. Hence $\nu L_s < 0{\cdot}0194\nu n^3$. For the present we shall regard νL_s as a variable parameter and discuss the solutions relative to this parameter.

In order to locate the zone of instability, we equate expression (27) to zero. Writing it in full, but omitting the term involving ν', which will be exceedingly small and will hardly affect the result, we find

$$\{(3-\nu L_s)\kappa^2+n^2\}\{2\nu L_s\kappa^2+n^2\}-4\kappa^2n^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This equation, regarded as involving an unknown quantity n^2/κ^2, is precisely the equation used by MAXWELL to determine the condition of stability of the ring of particles when unperturbed by any satellite. The condition of the reality of n^2/κ^2 leads to the upper limit for ν just quoted. In our problem we may take the unknown quantity as κ^2/n^2, and then assuming a value for νL_s, solve the equation. The values of κ (for differing values of n) will give the position of the zones of instability of a ring of particles of mass and number assumed. Or, conversely, taking a position of instability, as shown by telescopic observations of the ring, we may determine the corresponding value of νL_s, which establishes the order of value of the mass and number of particles at that point.

I have found that the latter process leads to no satisfactory result, and hence I do not record the work.

It is interesting to examine the meaning of the condition previously referred to, that the maximum instability is found when $(\alpha_{1,0}\Theta_{5,0}-n^4)$ is approximately zero. On referring again to equation (28), it is clear that the broadest zone of instability will be found, owing to the extreme smallness of the last member, when $(\Theta_{1,0}-n^2)(\Theta_{5,0}-n^2)-4\kappa^2n^2$ changes most slowly with κ. This will occur when the equation (29) has equal roots. Equal roots appear when, by the variation of the parameter νL_s, κ/n passes from real to imaginary values, or when*

$$\nu L_s = 0{\cdot}039.$$

This is the upper limit of the criterion previously quoted from MAXWELL, and would imply that all the particles were of such mass and number as to be on the border-line of instability.

When νL_s has this value, we find that

$$2\nu L_s(3-\nu L_s)\kappa^4-n^4 = 0\,;$$

or

$$\Theta_{5,0}\Theta_{1,0}-n^4 = 0.$$

* TISSERAND, *loc. cit.*, p. 183.

Whence, by (11),

$$a_{1,0}\Theta_{5,0}-n^4 = -\Theta_{5,0}\{\Sigma\Sigma a_{r,s}\Theta_{r,s}+\ldots\}.$$

This right-hand member is of the same order of value as the factor of $\sin 2\tau$ in (26). In this case, then, c may assume a high value. But it is noticeable that only at the limit of MAXWELL'S relation is great instability to be found.

When expression (27) has a value far from zero, either by virtue of the value of κ/n or the value of νL_s, it is clear from (25) that 2τ is imaginary and hence c is imaginary, the solution being stable.

It might be inferred from this that if values of νL_s were chosen such that MAXWELL'S relation were not fulfilled the effect of the satellite would be to stabilise what would otherwise be an unstable system. As pointed out already, however, the original equations and their solutions, as given here, simply give the motion of the particles in the vicinity of certain circles. In some cases the motion may be such that the particles depart rapidly from this zero circle; this we have termed instability. In other cases the solutions may indicate that the particles never move far from the zero circle; and this type of motion we have termed stable. But it is clear that if a small arbitrary displacement were given to each of the particles in the latter case, nothing in this paper precludes the possibility of their departure finally from the zero circle. That is, they may be again unstable. What we have found here is a series of orbits for the particles when subject to the attractions of Saturn, a satellite, and one another. Those in which the particles have large inequalities result in collisions with the neighbouring rings of particles and hence a complete departure from their former positions. Those which have no large inequalities and hence avoid collisions with neighbouring rings of particles may yet prove unstable when an arbitrary disturbance is further imposed upon them.

(ii) The particular integral.

In the expression (24) there appears a denominator of the form

$$(\Theta_{1,0}-m^2)(\Theta_{5,0}-m^2)-4\kappa^2m^2. \quad \ldots \ldots \ldots \quad (30)$$

Here m takes all positive integral values including zero. When the conditions are such, therefore, that expression (30) is approximately zero, the term in the particular integral will become very great and departure from the orbit will result. This expression is the same as (29), which, it has been pointed out, gives the positions of the unstable solutions of the complementary function. It may therefore be said that all the unstable positions are in the vicinity of the zero values of (30), and the following remarks apply equally to both parts of the solutions.

Referring to the form (29) it is seen that there are two variables, κ/n and νL_s. For a given value of νL_s there are in general two values of κ/n, and for a given value of κ/n there are two values of νL_s. In the figure (p. 125), the relation between κ/n and νL_s is shown graphically, only those values of νL_s which satisfy MAXWELL'S criterion being chosen. It will be seen that κ/n increases slowly from unity as νL_s increases from zero, until νL_s reaches the value 0·039. At this point the curve turns back and rises rapidly to an asymptote at $\nu L_s = 0$.

In the case when expression (30) is exactly zero, it is seen from elementary principles that the independent variable ϕ would appear explicitly. With passage of time, therefore, ρ and σ would increase linearly in magnitude and there would be complete departure of the particles from the vicinity of $r = a$.

§ 5. *Case where the Particles forming the Ring are of Unequal Masses.*

The previous equations (2) were reduced to the form (3) on the supposition that all the masses m_λ were of the same value m, $= \nu M$. We now proceed to the modifications introduced when these masses are all distinct in value.

Equations (2) with the same reductions as before, but maintaining the separate values m_λ, become:

$$\left.\begin{aligned}
\rho''_\lambda - 2\kappa\sigma'_\lambda = {} & \kappa^2\kappa'^{4/3}\nu'\left\{\frac{d}{d\alpha}\left(\tfrac{1}{2}b_0 + \ldots + b_i\cos i\phi + \ldots\right) - \cos\phi\right\} - \kappa^2 E_\lambda \\
& + \left[3\kappa^2 + \kappa^2\kappa'^2\nu'\frac{d^2}{d\alpha^2}\left(\tfrac{1}{2}b_0 + \ldots + b_i\cos i\phi + \ldots\right) - \kappa^2 F_\lambda\right]\rho_\lambda + \sum_\mu \kappa^2 G_{\mu,\lambda}\rho_\mu \\
& + \left[\kappa^2\kappa'^{4/3}\nu'\frac{d}{d\alpha}\left(b_1\sin\phi + \ldots + ib_i\sin i\phi + \ldots\right) - \kappa^2\kappa'^{4/3}\nu'\sin\phi - \kappa^2 H_\lambda\right]\sigma_\lambda \\
& + \sum_\mu \kappa^2 J_{\mu,\lambda}\sigma_\mu ; \\
\sigma''_\lambda + 2\kappa\rho'_\lambda = {} & \nu'\kappa^2\kappa'^{2/3}\left(b_1\sin\phi + \ldots + ib_i\sin i\phi + \ldots\right) - \nu'\kappa^2\kappa'^{4/3}\sin\phi + \nu\kappa^2 E'_\lambda \\
& + \Bigg[\nu'\kappa^2\kappa'^{4/3}\left(\ldots + i\frac{db_i}{d\alpha}\sin i\phi + \ldots\right) + \nu'\kappa^2\kappa'^{2/3}\left(\ldots + ib_i\sin i\phi \ldots\right) \\
& \qquad - \kappa^2 F'_\lambda\Bigg]\sigma_\lambda + \sum_\mu \kappa^2 G_{\mu,\lambda}\rho_\mu \\
& + \left[-\nu'\kappa^2\kappa'^{2/3}\left(\ldots + i^2 b_i\cos i\phi + \ldots\right) + \nu'\kappa^2\kappa'^2\cos\phi - \kappa^2 H'_\lambda\right]\sigma_\lambda \\
& + \sum_\mu \kappa^2 J'_{\mu,\lambda}\sigma_\mu .
\end{aligned}\right\} \quad (31)$$

In these equations

$$E_\lambda = \sum_\mu \frac{m_\mu}{M}\left\{\frac{1}{4\sin(\mu-\lambda)\pi/n} + \cos(\mu-\lambda)2\pi/n\right\},$$

$$F_\lambda = \sum_\mu \frac{m_\mu}{M}\left\{\frac{1}{8\sin^3(\mu-\lambda)\pi/n} - \frac{3}{\sin(\mu-\lambda)\pi/n}\right\},$$

$$G_{\mu,\lambda} = \frac{m_\mu}{M}\left\{\frac{\cos(\mu-\lambda)2\pi/n}{8\sin^3(\mu-\lambda)\pi/n} + \frac{\frac{3}{8}}{\sin(\mu-\lambda)\pi/n} + 2\cos(\mu-\lambda)2\pi/n\right\},$$

$$\mathrm{H}_\lambda = \sum_\mu \frac{m_\mu}{\mathrm{M}}\left\{\frac{\cos(\mu-\lambda)\,\pi/n}{8\sin^2(\mu-\lambda)\,\pi/n} + \sin(\mu-\lambda)\,2\pi/n\right\},$$

$$\mathrm{J}_{\mu,\lambda} = \frac{m_\mu}{\mathrm{M}}\left\{\frac{\cos(\mu-\lambda)\,\pi/n}{8\sin^2(\mu-\lambda)\,\pi/n} + \sin(\mu-\lambda)\,2\pi/n\right\},$$

$$\mathrm{E}'_\lambda = \sum_\mu \frac{m_\mu}{\mathrm{M}}\left\{\frac{\cos(\mu-\lambda)\,\pi/n}{4\sin^2(\mu-\lambda)\,\pi/n} - \sin(\mu-\lambda)\,2\pi/n\right\},$$

$$\mathrm{F}'_\lambda = \sum_\mu \frac{m_\mu}{\mathrm{M}}\left\{\frac{3\cos(\mu-\lambda)\,\pi/n}{8\sin^2(\mu-\lambda)\,\pi/n}\right\},$$

$$\mathrm{G}'_{\mu,\lambda} = \frac{m_\mu}{\mathrm{M}}\left\{-\frac{\cos(\mu-\lambda)\,\pi/n}{8\sin^2(\mu-\lambda)\,\pi/n} + 2\sin(\mu-\lambda)\,2\pi/n\right\},$$

$$\mathrm{H}'_\lambda = \sum_\mu \frac{m_\mu}{\mathrm{M}}\left\{\frac{\cos(\mu-\lambda)\,2\pi/n}{8\sin^3(\mu-\lambda)\,\pi/n} - \frac{\frac{3}{32}\sin^2(\mu-\lambda)\,2\pi/n}{\sin^5(\mu-\lambda)\,\pi/n} - \cos(\mu-\lambda)\,2\pi/n\right\},$$

$$\mathrm{J}'_{\mu,\lambda} = \frac{m_\mu}{\mathrm{M}}\left\{\frac{\cos(\mu-\lambda)\,2\pi/n}{8\sin^3(\mu-\lambda)\,\pi/n} - \frac{\frac{3}{32}\sin^2(\mu-\lambda)\,2\pi/n}{\sin^5(\mu-\lambda)\,\pi/n} - \cos(\mu-\lambda)\,2\pi/n\right\}. \quad . \quad . \quad (32)$$

These equations may be written, with a slightly different meaning of the quantities Θ from those formerly obtaining, in the form,

$$\left.\begin{aligned}
&\rho''_\lambda - 2\kappa\sigma'_\lambda + \rho_\lambda\sum_r\{\Theta_{1,r}\cos r\phi\} + \kappa^2\sum_\mu \mathrm{G}_{\mu,\lambda}\rho_\mu + \sigma_\lambda\sum_r\{\Theta_{2,r}\sin r\phi\} \\
&\qquad\qquad + \kappa^2\sum_\mu \mathrm{J}_{\mu,\lambda}\sigma_\mu = \sum_r \Theta_{3,r}\cos r\phi, \\
&\sigma''_\lambda + 2\kappa\rho'_\lambda + \rho_\lambda\sum_r\{\Theta_{4,r}\sin r\phi\} + \kappa^2\sum_\mu \mathrm{G}'_{\mu,\lambda}\rho_\mu + \sigma_\lambda\sum_r\{\Theta_{5,r}\cos r\phi\} \\
&\qquad\qquad + \kappa^2\sum_\mu \mathrm{J}'_{\mu,\lambda}\sigma_\mu = \sum_r \Theta_{6,r}\sin r\phi.
\end{aligned}\right\} \quad . \quad (33)$$

We shall determine the particular integral arising from one term of the right-hand member of the first equation, writing it typically $\frac{1}{2}\Theta_{3,m}e^{\iota m\phi}$.

Assume that

$$\rho_\lambda = \mathrm{A}_\lambda e^{\iota m\phi},$$

and

$$\sigma_\lambda = \mathrm{X}_\lambda e^{\iota m\phi}$$

for all values of λ.

Equations (33) then become

$$\left.\begin{aligned}
&-m^2\mathrm{A}_\lambda + 2\iota m\mathrm{A}'_\lambda + \mathrm{A}''_\lambda - 2\kappa(\iota m\mathrm{X}_\lambda + \mathrm{X}'_\lambda) + \mathrm{A}_\lambda\sum\{\Theta_{1,r}\cos r\phi\} \\
&\qquad + \kappa^2\sum \mathrm{A}_\mu \mathrm{G}_{\mu,\lambda} + \mathrm{X}_\lambda\sum\{\Theta_{2,r}\sin r\phi\} + \kappa^2\sum \mathrm{X}_\mu \mathrm{J}_{\mu,\lambda} = \tfrac{1}{2}\Theta_{3,m}, \\
&-m^2\mathrm{X}_\lambda + 2\iota m\mathrm{X}'_\lambda + \mathrm{X}''_\lambda + 2\kappa(m\mathrm{A}_\lambda + \mathrm{A}'_\lambda) + \mathrm{A}_\lambda\sum\{\Theta_{4,r}\sin r\phi\} \\
&\qquad + \kappa^2\sum \mathrm{A}_\mu \mathrm{G}'_{\mu,\lambda} + \mathrm{X}_\lambda\sum\{\Theta_{5,r}\cos r\phi\} + \kappa^2\sum \mathrm{X}_\mu \mathrm{J}'_{\mu,\lambda} = 0.
\end{aligned}\right\} \quad . \quad (34)$$

There are n pairs of equations in this form corresponding to the n values of λ.

As a solution we now take

$$A_\lambda = A_0^\lambda + \Sigma\Sigma A_{r,s}^\lambda \Theta_{r,s} + \quad . \quad . \quad . \quad . \quad ,$$
$$X_\lambda = X_0^\lambda + \Sigma\Sigma X_{r,s}^\lambda \Theta_{r,s} + \quad . \quad . \quad . \quad . \quad ,$$

with the same restrictions as before.

Substitute in (34) and equate to zero the terms involving no Θ but $\Theta_{1,0}{}^\lambda$ and $\Theta_{5,0}{}^\lambda$. Then, for all values of λ from 1 to n,

$$\left.\begin{aligned} -m^2 A_0^\lambda - 2\kappa\iota m X_0^\lambda + \Theta_{1,0}^\lambda A_0^\lambda + \kappa^2 \sum_\mu A_0^\mu G_{\mu,\lambda} + \kappa^2 \sum_\mu X_0^\mu J_{\mu,\lambda} &= \tfrac{1}{2}\Theta_{3,m}, \\ -m^2 X_0^\lambda + 2\kappa\iota m A_0^\lambda + \Theta_{5,0}^\lambda X_0^\lambda + \kappa^2 \sum_\mu A_0^\mu G'_{\mu,\lambda} + \kappa^2 \sum_\mu X_0^\mu J'_{\mu,\lambda} &= 0. \end{aligned}\right\} \quad . \quad . \quad (35)$$

These $2n$ equations can be solved by the usual processes to give the values of the constants A_0^λ and X_0^λ. It is not necessary for us to work out the results in detail, it is sufficient to note that the determinant of the left-hand members will appear as the denominator in each case. The determinant is the following:

$$\left.\begin{vmatrix} -m^2+\Theta_{1,0}^1, & \kappa^2 G_{2,1}, & \kappa^2 G_{3,1}, \ldots, & \kappa^2 G_{n,1}; & -2\kappa\iota m, & \kappa^2 J_{2,1}, & \kappa^2 J_{3,1}, \ldots, & \kappa^2 J_{n,1} \\ 2\kappa\iota m, & \kappa^2 G'_{2,1}, & \kappa^2 G'_{3,1}, \ldots, & \kappa^2 G'_{n,1}; & -m^2+\Theta_{5,0}^1, & \kappa^2 J'_{2,1}, & \kappa^2 J'_{3,1}, \ldots, & \kappa^2 J'_{n,1} \\ \kappa^2 G_{1,2}, & -m^2+\Theta_{1,0}^2, & \kappa^2 G_{3,2}, \ldots, & \kappa^2 G_{n,2}; & \kappa^2 J_{1,2}, & -2\kappa\iota m, & \kappa^2 J_{3,2}, \ldots, & \kappa^2 J_{n,2} \\ \kappa^2 G'_{1,2}, & 2\kappa\iota m, & \kappa^2 G'_{3,2}, \ldots, & \kappa^2 G'_{n,2}; & \kappa^2 J'_{1,2}, & -m^2+\Theta_{5,0}^2, & \kappa^2 J'_{3,2}, \ldots, & \kappa^2 J'_{n,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \kappa^2 G_{1,n}, & \kappa^2 G_{2,n}, & \kappa^2 G_{3,n}, \ldots, & -m^2+\Theta_{1,0}^n; & \kappa^2 J_{1,n}, & \kappa^2 J_{2,n}, & \kappa^2 J_{3,n}, \ldots, & -2\kappa\iota m \\ \kappa^2 G'_{1,n}, & \kappa^2 G'_{2,n}, & \kappa^2 G_{3,n}, \ldots, & 2\kappa\iota m; & \kappa^2 J'_{1,n}, & \kappa^2 J'_{2,n}, & \kappa^2 J'_{3,n}, \ldots, & -m^2+\Theta_{5,0}^n \end{vmatrix}\right\}. \quad (36)$$

This determinant corresponds to the denominators in expressions (24). When it vanishes or becomes small, it is clear, as before, that the terms of the solution tend to become large, and instability follows.

In estimating the values of F, G, H and J, which appear in the above determinant, it is to be noted that m_μ/M is exceedingly small for all values of μ. But the quantities in which it appears may be large by virtue of the small denominators which are involved. In the expression for F_λ, the term $3/\sin(\mu-\lambda)\pi/n$ may be neglected in comparison with the first term for large values of n. Also, $\sum_\mu \frac{m_\mu}{M} \frac{1}{8\sin^3(\mu-\lambda)\pi/n}$ will lie between zero and $\frac{\overline{m}}{M} \Sigma \frac{1}{8\sin^3(\mu-\lambda)\pi/n}$ since all the signs are positive, if $\overline{m}$ is the greatest value of m_μ appearing in the ring. Hence F_λ lies between zero and $0{\cdot}0096\, n^3\overline{m}/M$ in value.

In the same way the value of $G_{\mu,\lambda}$ will arise almost wholly from the first term. The largest value it may have will be $\overline{m}n^3/8\pi^3M$ or $0{\cdot}004 n^3\overline{m}/M$. E_λ, H_λ, $J_{\mu,\lambda}$, E'_λ, F'_λ and $G'_{\mu,\lambda}$ are seen to be one order lower in the reciprocal of $\sin(\mu-\lambda)\pi/n$ and therefore may be neglected. H'_λ has the limit $-0{\cdot}0192 n^3\overline{m}/M$, and $J'_{\mu,\lambda}$ the limit $-0{\cdot}008 n^3\overline{m}/M$.

We shall assume that the number of particles in any ring is large. It is probable that they vary in magnitude from the infinitesimally small up to the limit given by MAXWELL. Hence the values of the expressions F_λ, $G_{\mu,\lambda}$, H'_λ and J'_λ will vary over a range of values, between the given limits, as λ takes its successive values.

Reverting to determinant (36), we see that it may now be written

$$\left|\begin{array}{ccccc|ccccc} -m^2+\Theta^1_{1,0}, & \kappa^2 G_{2,1}, & \kappa^2 G_{3,1}, & \ldots, & \kappa^2 G_{n,1}; & -2\kappa\iota m, & 0, & 0, & \ldots, & 0 \\ 2\kappa\iota m, & 0, & 0, & \ldots, & 0; & -m^2+\Theta^1_{5,0}, & \kappa^2 J'_{2,1}, & \kappa^2 J'_{3,1}, & \ldots, & \kappa^2 J'_{n,1} \\ \kappa^2 G_{1,2}, & -m^2+\Theta^2_{1,0}, & \kappa^2 G_{3,2}, & \ldots, & \kappa^2 G_{n,2}; & 0, & -2\kappa\iota m, & 0, & \ldots, & 0 \\ 0, & 2\kappa\iota m, & 0, & \ldots, & 0; & \kappa^2 J'_{1,2}, & -m^2+\Theta^2_{5,0}, & \kappa^2 J'_{3,2}, & \ldots, & \kappa^2 J'_{n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \kappa^2 G_{1,n}, & \kappa^2 G_{2,n}, & \kappa^2 G_{3,n}, & \ldots, & -m^2+\Theta^n_{1,0}; & 0, & 0, & 0, & \ldots, & -2\kappa\iota m \\ 0, & 0, & 0, & \ldots, & 2\kappa\iota m; & \kappa^2 J'_{1,n}, & \kappa^2 J'_{2,n}, & \kappa^2 J'_{3,n}, & \ldots, & -m^2+\Theta^n_{5,0} \end{array}\right| \quad (37)$$

For all conditions satisfying MAXWELL's criterion, the quantities $G_{\lambda,\mu}$, $J'_{\lambda,\mu}$ will be small. So that, provided κ^2 is not too great, the value of the determinant (37) will be small for those values of κ that satisfy the relation

$$\left|\begin{array}{cccccccc} -m^2+\Theta^1_{1,0}, & 0, & \ldots, & 0, & -2\kappa\iota m, & 0, & \ldots, & 0 \\ 2\kappa\iota m, & 0, & \ldots, & 0, & -m^2+\Theta^1_{5,0}, & 0, & \ldots, & 0 \\ 0, & -m^2+\Theta^2_{1,0}, & \ldots, & 0, & 0, & -2\kappa\iota m, & \ldots, & 0 \\ 0, & 2\kappa\iota m, & \ldots, & 0, & 0, & -m^2+\Theta^2_{5,0}, & \ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & -m^2+\Theta^n_{1,0}, & 0, & 0, & \ldots, & 2\kappa\iota m \\ 0, & 0, & \ldots, & 2\kappa\iota m, & 0, & 0, & \ldots, & -m^2+\Theta^n_{5,0} \end{array}\right| = 0. \quad (38)$$

This relation is satisfied by those values of κ which satisfy the equation

$$(-m^2+\Theta^\lambda_{1,0})(-m^2+\Theta^\lambda_{5,0})-4\kappa^2 m^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where λ takes all its integral values in turn. Further it is easily shown that, on any distribution with n large, $-F_\lambda = \frac{1}{2}H'_\lambda$. Hence we fall back upon the same type of equation as we had in the case of equal particles (equation (29)) where we replace $_\nu L_s$ by F_λ.

Instead of treating the equation (39) separately for the various integral values of λ, since n is large, we may imagine a single equation with the assumption that F_λ is an arbitrary variable parameter. The determinant (37) will then be small, and instability result for all the values of κ given by (39), for all values of the parameter F_λ that exist. With a wide range of values of F_λ corresponding to a wide range in the magnitudes of the masses of the particles, we may expect to find a broad region of instability.

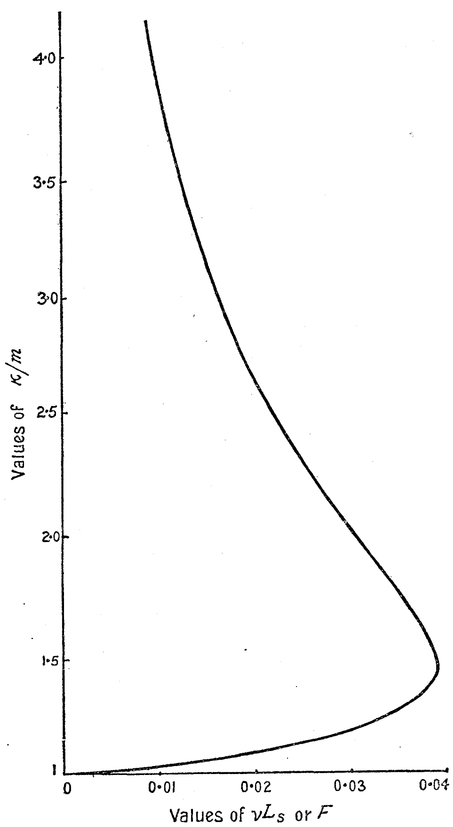
It can readily be shown that the condition (39) would also be produced if the general case of unequal particles were solved for the complementary function in the same way as has been done for the case of equal particles, which produced (29). This work is not reproduced owing to the length and complexity of the expressions, and also because the results are wholly contained in the condition (39) produced from the particular integral.

§ 6. *Application of the Results to the Saturnian System.*

Equation (39) written out in full is

$$\{\kappa^2(3-F_\lambda)+m^2\}\{2F_\lambda\kappa^2+m^2\}-4\kappa^2m^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

In this equation m is any integer and F_λ may vary between zero and $0{\cdot}0096n^3\bar{m}/M$. As the distinctions indicated by the suffix λ are now of no importance, it may be dropped. The solutions of (40) will give approximately the positions where divisions in the Ring of Saturn may be expected.



For any given value of F, there are four values of κ/m, two pairs equal with opposite signs. For any given value of κ/m there are two values of F; one, however, being greater than the MAXWELL limit, is excluded. The limiting value of F for real values of κ/m is 0·039. This is, of course, the same result as that found by MAXWELL.

The relation between κ/m and F is shown in the figure, and the table shows actual numerical values.

We may readily assume that in the existing Rings of Saturn there are particles of all masses from the infinitesimal to MAXWELL's upper limit. These will give rise to varying values of F, depending upon the masses of the particles adjacent to the particle under consideration. The maximum value of F is itself small compared with unity; we shall then arrive at a limit of κ by taking $F = 0$ in equation (40). We find thus that the boundary of a division should occur at $\kappa/m = 1$, for each integral value of m.

νL_8 or F.	κ/m.		$m = 1$.		$m = 2$.
			a/a'.		a/a'.
0	1	∞	0	1	0·6299
0·01	1·0395	3·9336	0·0243	0·8223	0·6456
0·015	1·0639	3·1412	0·1534	0·7745	0·6549
0·020	1·0940	2·6509	0·2102	0·7349	0·6656
0·030	1·1750	2·0156	0·2810	0·6333	. . . . .
0·038	1·3353	1·5772	0·3981	0·5117	
0·039	1·4424	—	0·4549		

Remembering that $\kappa = -\omega(\omega'-\omega)^{-1}$, $\omega^2 a^3 = \omega'^2 a^3$, we find:

$$\begin{aligned} \text{For } m &= 1, & \omega &= \infty, & a/a' &= 0, \\ m &= 2, & \omega/\omega' &= 2\ , & a/a' &= 0{\cdot}62996, \\ m &= 3, & \omega/\omega' &= \tfrac{3}{2}\ , & a/a' &= 0{\cdot}76289, \\ m &= 4, & \omega/\omega' &= \tfrac{4}{3}\ , & a/a' &= 0{\cdot}82524. \end{aligned}$$

[It should be remarked that a positive value of κ gives positions without the satellite orbit, and a negative value of κ gives positions within. As κ appears in (40) in the form of a square, both positive and negative forms result. We should therefore have the same phenomena in a ring of particles beyond the satellite orbit as we find within].

The result $a = 0$ implies a division of the ring at the origin. This would fall within the planet itself. But if the zone consequent upon the variation of F is extensive, it may extend beyond the surface of the planet and show a clearance of particles there.

For $m = 2$, $a/a' = 0{\cdot}62996$. In the case of satellite Mimas this should indicate the commencement of a division in the ring at distance 16·9″. CASSINI's Division begins at 16·87″ and ends at 17·64″. This agreement is very remarkable.

Reference to the figure shows that in the vicinity of $\kappa/m = 1$, κ increases with F. But as κ increases so does a/a'. Hence the instability caused by the larger values of F should be in positions corresponding to larger values of κ, that is, to larger values of a/a'. In other words, the division should extend outwards. This agrees with the observational data just quoted. We may then attribute the production of CASSINI's Division to Mimas.

For $n = 3$, $a/a' = 0{\cdot}76289$.

For satellite Mimas, this should cause a division at distance 20·46″. This is just beyond the outer edge of Ring A, which terminates at 20·01″.

Considering next the satellite Enceladus, we should find a division at the origin for $n = 1$, and at distance 21·69″ for $n = 2$. The last is again just beyond the limits of Ring A.

The remaining satellites all produce instability at the origin, but the other points at which this occurs are outside the existing ring.

We may use the observations of the dimensions of Cassini's Division to determine the maximum value of F appearing. As we have already found, the inner edge corresponds closely to $F = 0$. The radius of the outer edge is $a = 17{\cdot}64''$. Hence for satellite Mimas $a/a' = 0{\cdot}65781$, giving $\kappa/m = 1{\cdot}0720$.

If now equation (40) be solved for F, taking this value for κ/m, the result is $F = 0{\cdot}0173$. Hence we may conclude that F ranges from zero to 0·0173.

Using this value of F, we now proceed to the study of the roots of equation (40). Solving, we find

$$\kappa/m = \pm 1{\cdot}0720 \quad \text{and} \quad \kappa/m = \pm 2{\cdot}8917.$$

Take $m = 1$. Then

$$a/a' = 0{\cdot}1712 \quad \text{and} \quad a/a' = 0{\cdot}7535.$$

We may expect to find a clearance of particles from $a/a' = 0$ to 0·1712; and from $a/a' = 0{\cdot}7535$ to unity.

The first gives the extent of the clearance near the origin.

For the various satellites its dimensions are:

Mimas	$a = 4{\cdot}59''$,
Enceladus	$a = 6{\cdot}16''$,
Tethys	$a = 7{\cdot}30''$,
Dioné	$a = 9{\cdot}34''$,
Rhea	$a = 13{\cdot}07''$,
Titan	$a = 29{\cdot}94''$.

The radius $a = 9{\cdot}34''$ indicates approximately the inner radius of the Crêpe Ring, while $a = 13{\cdot}07''$ indicates more closely the inner radius of Ring B.

Applying the second ratio, $a/a' = 0{\cdot}7535$, to Mimas, we find radius $a = 20{\cdot}2''$. There should be a clearance of particles from 20·2″ up to the satellite itself. This indicates with considerable precision the termination of the whole ring, which has a radius 20·01″.

These results are subject to modification owing to the effect of the oblateness of the planet Saturn and the influence of one ring upon another. But the agreement of theory and observation in this first approximation is sufficiently remarkable.

The interpretation of the effect of Dioné and Rhea on the inner parts of the ring is not clear. From the theory one would expect that any satellite could affect a clearance of particles from the origin up to a radius given by $a/a' = 0{\cdot}1712$. In that case Titan, the largest of the satellites, should dissipate the whole of the existing rings, for this ratio carries us far beyond the outer radius.

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There are therefore two facts to explain. First, the existence of the Crêpe Ring within the dissipative area of Rhea, and second, the existence of the bright rings within the dissipative area of Titan. In connection with the first, LOWELL has noted a definite black band within Ring B, so that there is a clearance of particles between the Crêpe Ring and the bright rings. It would appear as though the dissipative power of the satellites was only effective near the outer boundary of the unstable area about the origin. To discuss this, let as examine the analytical results.

It has already been pointed out how very small the exponent c is, as given by (18), indicating a very slow rate of dispersion. Consider, instead, the numerators of the expressions (24), the vanishing of the denominators of which causes the instability. The numerators are small because of the quantities $\Theta_{3,m}$ and $\Theta_{6,m}$. In the case under discussion, $m = 1$. From (5)

$$\Theta_{3,1} = \nu'\kappa^2\kappa'^{4/3}\left\{\frac{db_1}{d\alpha}-1\right\},$$

$$\Theta_{6,1} = \nu'\kappa^2\left\{\kappa'^{2/3}b_1-\kappa'^{4/3}\right\}.$$

Using the well-known expression for b_1,* we find

$$\Theta_{3,1} = \nu'\kappa^2\kappa'^{4/3}\left\{\tfrac{9}{8}\alpha^2+\tfrac{225}{192}\alpha^4+\ldots\right\}$$

$$= \nu'\left\{\omega'/\omega-1\right\}^{-2}\left\{\tfrac{9}{8}\alpha^4+\tfrac{225}{192}\alpha^6+\ldots\right\}$$

$$\Theta_{6,1} = \nu'\kappa^2\left\{\alpha\left(\alpha+\tfrac{3}{8}\alpha^3+\tfrac{45}{192}\alpha^5\ldots\right)-\alpha^2\right\}$$

$$= \nu'\left\{\omega'/\omega-1\right\}^{-2}\left\{\tfrac{3}{8}\alpha^4+\tfrac{45}{192}\alpha^6\ldots\right\}.$$

For small values of a, ω'/ω is small, and the value of $(\omega'/\omega-1)^{-2}$ will be greater than, but not far from, unity. Hence the values of $\Theta_{3,1}$ and $\Theta_{6,1}$ depend approximately upon the fourth power of α or a/a'. It is clear then that the numerators in (24) will be vanishingly small except for the larger values of a/a'.

The physical meaning is that, while instability will always take place when the denominators vanish, the rate of dissipation will be small except for the largest values of α which are permissible. There will also be a uniform grading in the rate of dissipation as α increases.

Applying this result to the case of Saturn's satellites, we may expect to find actually a clearance only near the outer limits of the areas under consideration. The areas of clearance of the first three satellites fall within the body of the planet. Dioné causes the clearance between the surface of the planet at 8·65″ and 9·34″, which is approximately the commencement of the Crêpe Ring. The limit of the area of clearance of Rhea is 13·07″, and only near that boundary is the action effective, the Crêpe Ring being undispersed in the weaker part of the field. The bright rings are clearly in the weak part of Titan's field of clearance, and so continue to exist. It is obvious, however, that with passage of time the Crêpe Ring will be dispersed by Rhea and the whole by Titan.

* TISSERAND, vol. i., p. 272.

We have found the maximum value of F appearing as 0·0173. It was previously shown that the limiting value of F was $0·0096n^3\overline{m}/\mathrm{M}$. Hence

$$0·0096n^3\overline{m}/\mathrm{M} = 0·0173,$$

which gives

$$\overline{m}/\mathrm{M} = \frac{0·0173}{0·0096n^3}$$
$$= 1·8/n^3.$$

That is, the size of the largest particles is just below that given by MAXWELL'S criterion.

§ 7. *Summary and Conclusion.*

(1) Assuming that a planet is surrounded by concentric rings of particles performing approximately circular orbits when unperturbed, and that the influence of one ring upon another may be neglected to a first approximation, the effect upon these rings of a satellite performing also an unperturbed circular orbit is discussed.

If the particles in the rings are all equal, it is shown that we should expect, in certain places, large perturbations to take place, such that the particles in a particular ring would leave that ring and mingle with those of other rings, and so leave a "division."

(2) As there is no reason to believe that the particles in any ring are all equal, the analysis is extended to cover the case of unequal particles.

We assume that in any ring the number of particles is large, and that therefore we shall probably have one specimen at least of all particles from the smallest to the largest.

It is then shown that the divisions would become more extended, and therefore more readily visible.

(3) On the supposition that some of the particles at any rate are indefinitely small, we obtain CASSINI'S Division at once. On making use of the dimensions of this division to estimate the greatest magnitude of the particles in any ring, we find the following results:—

Satellite Mimas should produce a clearance of particles from radius 20·2″ up to itself. The ring should therefore terminate at 20·2″. Observation shows that it terminates at 20·01″.

Satellite Mimas should produce a division from radius 16·9″ to 17·64″. (This last measurement was used as a datum for estimating the magnitude of the greatest particles.) Observation gives the limits of CASSINI'S Division as 16·87″ and 17·64″.

Satellite Dioné should produce a clearance of particles from the region of the surface of the planet up to radius 9·34″. The Crêpe Ring is observed to begin with a diffused edge at 10·83″.

Satellite Rhea should also produce a clearance of particles up to radius 13·07″. The inner edge of Ring B is observed to commence at 13·21″.

The existence of the Crêpe Ring in a dissipative area is also discussed.

(4) By the inclusion of the effect of the oblateness of Saturn and the influence of one ring of particles upon another these results might be still further improved.

The theory presented therefore gives a closely quantitative account of the salient features of Saturn's Ring. The numerous smaller divisions observed by LOWELL and others are not accounted for; but, for the reasons given in § 4, their existence is not excluded.

(5) The dimensions of CASSINI'S Division show that particles of all sizes up to a limit just short of that imposed by MAXWELL for stability exist in the rings.

Appendix on the Data of the Problem.

I. Dimensions and divisions of the ring in seconds of arc at mean distance*:—

Distance from centre of planet to—

Inner edge of Crêpe Ring		10·83″
Inner edge of Ring B		13·00″
Divisions of Ring B	B1	13·39″
	B2	14·04″
	B3	14·74″
	B4	15·32″
	B5	15·69″
	B6	15·95″
Outer edge of Ring B		16·87″
Inner edge of Ring A		17·64″
Division in Ring A		19·00″
Outer edge of Ring A		20·01″

II. Equatorial diameter of Saturn . . . . 17·30″

III. Elements of satellites:—

	Mean distance.	Mass as fraction of Saturn.
Mimas	26·82″	$7 \cdot 10^{-8}$
Enceladus	34·43″	$25 \cdot 10^{-8}$
Tethys	42·66″	$11 \cdot 10^{-7}$
Dioné	54·59″	$18{\cdot}7 \cdot 10^{-7}$
Rhea	76·38″	$4 \cdot 10^{-6}$
Titan	174·8″	$2{\cdot}1 \cdot 10^{-4}$

* LOWELL, 'Observatory Bulletin,' No. 68, and "Lecture" on April 26, 1916, in 'Journal of Royal Astron. Soc. of Canada.'